Statistical Mechanics - Problem set 3

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1. **Weak vs. Strong coupling:** Consider two qubits with

$$H_A = \frac{\Omega}{2}\sigma_z^A, \qquad H_B = \frac{\Omega}{2}\sigma_z^B. \tag{1}$$

Each qubit is first prepared in a thermal equilibrium state at temperatures T_A and T_B, so that at $t = 0$ the global state is

$$\rho(0) = \frac{e^{-\beta_A H_A}}{Z_A}\frac{e^{-\beta_B H_B}}{Z_B}. \tag{2}$$

Then at $t = 0$ they are put in contact and allowed to interact unitarily under the Hamiltonian

$$H = H_A + H_B + V, \tag{3}$$

where

$$V = g\sigma_x^A\sigma_x^B, \tag{4}$$

and g is a constant.

(a) Study how the average energy of each part evolves in time. In particular, recall that energy may be retained in the interaction, so that not all the energy leaving A will necessarily enter B. Play with the parameters and try to understand under which conditions the energy trapped in the interaction becomes negligible. That is, under which conditions we can say that indeed we are operating in the *weak-coupling regime*.

(b) Writing $\sigma_x = \sigma_+ + \sigma_-$ we may recast the interaction V in Eq. (4) in the form

$$V = g(\sigma_+^A\sigma_-^B + \sigma_-^A\sigma_+^B) + g(\sigma_+^A\sigma_+^B + \sigma_-^A\sigma_-^B). \tag{5}$$

Using time-dependent perturbation theory it is possible to show that in the weak-coupling regime the last term becomes negligible (don't worry, you don't need to show that here). This is called the **Rotating Wave Approximation (RWA)**. Compare your simulations in (a) with those obtained using only

$$\tilde{V} = g(\sigma_+^A\sigma_-^B + \sigma_-^A\sigma_+^B), \tag{6}$$

and convince yourself that, indeed, the RWA works.

2. **Carnot cycle for a harmonic oscillator:**[1] Imagine that a harmonic oscillator represents a trap for some gas. If we increase the frequency ω we compress the gas and if we decrease ω we expand it. Thus, for a harmonic oscillator, $1/\omega$ plays the same role as the volume in the case of a gas in a piston. In this problem the goal is to construct a Carnot cycle for a single harmonic oscillator. The Carnot cycle is described by the following strokes (see Fig. 1)

[1] This problem seems a bit long, but don't worry. It is pretty easy. :)

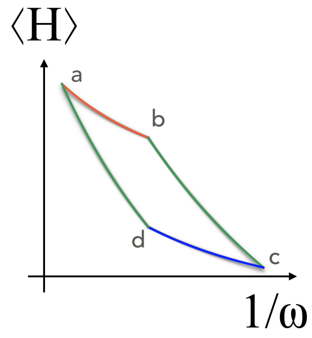

Figure 1: Example of a Carnot cycle for a single harmonic oscillator.

- *ab*: Hot isothermal expansion, where the frequency goes down ($\omega_b < \omega_a$) with the system always in contact with a hot bath at a temperature T_H.
- *bc*: Isentropic/adiabatic expansion, where the frequency goes down again ($\omega_c < \omega_a$) but without a contact to any bath.
- *cd*: Cold isothermal compression, where the frequency goes up ($\omega_d > \omega_c$) with the system always in contact with a cold bath at a temperature T_C.
- *da*: Isentropic compression, where the frequency goes up again ($\omega_a > \omega_d$) but without a contact to any bath.

I want you to study only a quasi-static cycle. So the idea is that both ω and T are changing with time, but we don't need to know the precise time dependence, because everything is quasi-static. All you need to know are the following facts. First, when the system is in contact with a bath at a temperature T, whatever its frequency may be, its energy will be given by

$$\langle H \rangle = \frac{\omega}{2} \coth\left(\frac{\omega}{2T}\right), \tag{7}$$

whereas the free energy is given by

$$F = T \ln \sinh\left(\frac{\omega}{2T}\right) - T \ln 2. \tag{8}$$

The second thing we need to know is what happens in the isentropic strokes, which are the strokes in which the dynamics is unitary (there is no heat bath). In this case we have to invoke the *adiabatic theorem* of quantum mechanics. According to this theorem, if we start with energy $\langle H \rangle_i$ and frequency ω_i, then if we slowly change the frequency to ω_f the final energy will be

$$\langle H \rangle_f = \frac{\omega_f}{\omega_i} \langle H \rangle_i. \tag{9}$$

Knowing only these facts we can compute everything for the quasi-static Carnot cycle. Your parameters are only T_H, T_C, ω_a, ω_b, ω_c and ω_d.

(a) Compute the heat and work in each of the strokes.

(b) Compute the efficiency,

$$\eta = \frac{|W|}{|Q_H|}, \tag{10}$$

where W is the total work done in the cycle and Q_H is the heat absorbed from the hot bath. In particular, show that the efficiency will in general be lower than the Carnot efficiency that we learn in thermodynamics

$$\eta_C = 1 - \frac{T_C}{T_H}. \tag{11}$$

(c) Find out what are the conditions in the T's and ω's to recover the Carnot efficiency.

3. **Statistics for multiple heat bath interactions:** Consider 3 qubits, which I will call A, B and C. They are all three prepared in thermal equilibrium at temperatures T_A, T_B and T_C and Hamiltonians

$$H_i = \frac{\Omega}{2}\sigma_z^i, \qquad i \in \{A, B, C\}. \tag{12}$$

At $t = 0$ we measure the three qubits in the σ_z basis. We then put them to interact sequentially. First, we put A to interact with B, by means of the thermal operation

$$V_{AB} = g(\sigma_+^A\sigma_-^B + \sigma_-^A\sigma_+^B), \tag{13}$$

which lasts for a time τ. Then we uncouple them, store B in a safe place, and put A to interact with C. The interaction is of the same form

$$V_{AC} = g(\sigma_+^A\sigma_-^C + \sigma_-^A\sigma_+^C), \tag{14}$$

and also lasts for a time τ. At the end of this process we measure A, B and C again in the same basis. Study the statistics of this problem (i.e., the probability of the corresponding quantum trajectory). Try to find some interesting features in this distribution. In particular, note that since B and C both interacted with A, the outcomes of B and C will not be statistically independent. Is there anything interesting in their correlations?