Quantum Information and Quantum Noise - Problem set 3

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1. **Measurements with two qubits.** Suppose you have a qubit S prepared in a certain state $|\psi\rangle$ which you want to measure. In this problem you will analyze one specific way of doing, by directly coupling S to another qubit ancilla A. We assume the ancilla is initialized in an arbitrary state

$$|\chi\rangle_A = \cos(\chi)|0\rangle_A + \sin(\chi)|1\rangle_A.$$

Then we evolve the composite state $|\psi\rangle_S \otimes |\chi\rangle_A$ with a CNOT

$$U = |0\rangle_S\langle 0| \otimes \mathbb{I}_A + |1\rangle_S\langle 1| \otimes \sigma_x^A.$$

Finally, after the evolution we perform a projective measurement on the ancilla in a certain basis:

$$|\phi_0\rangle_A = \cos(\phi)|0\rangle_A + \sin(\phi)|1\rangle_A,$$

$$|\phi_1\rangle_A = -\sin(\phi)|0\rangle_A + \cos(\phi)|1\rangle_A.$$

   (a) Find the set of generalized measurement operators $\{M_i\}$ associated with this protocol.

   (b) Find the associated POVM $\{E_i\}$.

   (c) Discuss for which values of ϕ and χ we get a projective measurement.

   (d) Discuss for which values of ϕ and χ we do no measurement at all. That is, such that $E_0 = E_1 = \mathbb{I}/2$.

2. **Single-mode squeezing.** The squeezing operator is defined as[1]

$$S_z = \exp\left\{\frac{1}{2}(za^\dagger a^\dagger - z^* aa)\right\},$$

where $z = re^{i\theta}$ is a complex number. Note that, by construction, S_z is unitary.

   (a) Show that

$$S_z^\dagger a S_z = a\cosh r + a^\dagger e^{i\theta}\sinh r.$$

   (b) Let $|z\rangle = S_z|0\rangle$ denote the *squeezed vacuum*. Find the coefficients of this state in the Fock basis. Tip: use the following result[2]

$$S(z) = e^{\frac{a^\dagger a^\dagger}{2} e^{i\theta}\tanh(r)} e^{-\ln(\cosh(r))(a^\dagger a + 1/2)} e^{-\frac{aa}{2} e^{-i\theta}\tanh(r)}. \tag{1}$$

3. **Eigenstates of position and momentum** We can use squeezing to connect coherent states with the eigenstates of the position and momentum operators (also called quadratures in quantum optics):

$$\hat{q} = \frac{a + a^\dagger}{\sqrt{2}}, \qquad \hat{p} = \frac{i}{\sqrt{2}}(a^\dagger - a). \tag{2}$$

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[1] Some people define it with $z \to -z$.

[2] For a proof of Eqs. (1) and (11), see Appendix B of B. Schumaker and C. Caves, *Phys. Rev. A.*, **31**, 3093 (1985).

Their eigenstuff are defined as $\hat{q}|q\rangle = q|q\rangle$ and $\hat{p}|p\rangle = p|p\rangle$. To do that, we define the *displaced squeezed state*

$$|\alpha, z\rangle = D_\alpha S_z|0\rangle, \tag{3}$$

where $D_\alpha = e^{\alpha a^\dagger - \alpha^* a}$ is the displacement operator.

(a) Verify that

$$a|\alpha, z\rangle = \alpha|\alpha, z\rangle + e^{i\theta}\sinh(r) D_\alpha S_z a^\dagger|0\rangle, \tag{4}$$

$$a^\dagger|\alpha, z\rangle = \alpha^*|\alpha, z\rangle + \cosh(r) D_\alpha S_z a^\dagger|0\rangle. \tag{5}$$

(b) Use these results to show that $|\alpha, z\rangle$ tends to the eigenstates of $\hat{q}$ in the limit of infinite squeezing, $r \to \infty$, provided we choose $\theta = \pi$. Similarly, show that the eigenstates of $\hat{p}$ are obtained when $\theta = 0$. What are the corresponding eigenvalues? This provides a clear interpretation for the real and imaginary parts of α in a coherent state.

4. **Two-mode squeezing.** Consider two bosonic modes a and b. We define the two-mode squeezing operator as

$$\mathcal{T}_z = \exp\left\{z a^\dagger b^\dagger - z^* ab\right\}, \qquad z = re^{i\theta} \tag{6}$$

Two-mode squeezing is an entangling unitary. In fact, in the context of continuous variables, this is the most widely used way to entangle two modes.

(a) Show that

$$\mathcal{T}_z^\dagger a \mathcal{T}_z = a\cosh(r) + b^\dagger e^{i\theta}\sinh(r) \tag{7}$$

$$\mathcal{T}_z^\dagger b \mathcal{T}_z = b\cosh(r) + a^\dagger e^{i\theta}\sinh(r) \tag{8}$$

(b) Suppose now that a and b are prepared in a two-mode squeezed state,

$$|\psi\rangle = \mathcal{T}_z|0, 0\rangle. \tag{9}$$

Compute the reduced density matrix of a and b and show that this will have the form of a thermal state,

$$\rho_{\text{th}} = (1 - e^{-\beta})\sum_{n=0}^{\infty} e^{-\beta n}|n\rangle\langle n|, \tag{10}$$

where $\beta > 0$ represents the inverse temperature. Tip: use the decomposition

$$\mathcal{T}_z = e^{a^\dagger b^\dagger e^{i\theta}\tanh(r)} e^{-\ln(\cosh(r))(a^\dagger a + b^\dagger b + 1)} e^{-ab e^{-i\theta}\tanh(r)}. \tag{11}$$

Show that in this case β is associated with the degree of two-mode squeezing z. Thus, we can think about a thermal state of a single mode as actually being a pure state of a larger system. This appears often in quantum field theory (search for “ER = EPR").