Statistical Mechanics - 2019-2 - Problem set 4

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Deadline: 26/11

1. **Exchange Hamiltonian:** Consider a system described by two modes a and b (can be either bosonic or fermionic) with Hamiltonian

$$\mathcal{H} = \epsilon_a a^\dagger a + \epsilon_b b^\dagger b - J(a^\dagger b + b^\dagger a). \tag{1}$$

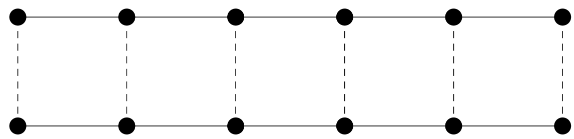
Diagonalize this Hamiltonian using the general recipe for quadratic Hamiltonians discussed in class.

2. **Tight-binding ladder:** Consider two 1D tight-binding chains disposed in the form of a ladder, as in the figure below. We assume the system to be bosonic and described by a set of operators a_i for the upper ladder and b_i for the lower ladder, where $i = 1, 2, \ldots, L$.

$$\mathcal{H} = \sum_{i=1}^{L} \Big(\epsilon_a a_i^\dagger a_i + \epsilon_b b_i^\dagger b_i\Big) - g_a \sum_{i=1}^{L} (a_i^\dagger a_{i+1} + a_{i+1}^\dagger a_i) - g_b \sum_{i=1}^{L} (b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i)$$
$$- J \sum_{i=1}^{L} (a_i^\dagger b_i + b_i^\dagger a_i).$$

Assume periodic boundary conditions.

(a) Diagonalize this Hamiltonian. What I recommend is to first move to momentum space for each lattice individually and then try to use the results of problem 1.



(b) You will find that the dispersion relation is composed of two energy bands. Analyze the shape of these bands. For simplicity you may focus on $\epsilon_a = \epsilon_b$ and set $J = 1$ (which defines the energy scale). Play with the different choices of g_a and g_b.

3. **Fermionic triple well:** Consider a lattice containing only 3 sites, populated by *spinless* fermions evolving according to Hamiltonian

$$\mathcal{H} = U \sum_i \hat{n}_i \hat{n}_{i+1} - g \sum_i (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i), \tag{2}$$

where we are assuming periodic boundary conditions. The first term describes a nearest-neighbor interaction, whereas the second is the usual hopping term. Suppose the lattice is populated with exactly 2 fermions.

(a) List the possible Fock states. Notice that states with two Fermions have a sign ambiguity. For instance, you can define $c_1^\dagger c_2^\dagger |0\rangle$ or $c_2^\dagger c_1^\dagger |0\rangle$. The two differ by a minus sign. You need to choose one representation and then be consistent with it throughout.

(b) Write down the matrix elements of $\mathcal{H}$ in this basis.

(c) Find the eigenvalues and eigenvectors of $\mathcal{H}$.

4. **Bogoliubov transformation:** Consider a Fermionic system described by two operators a and b, with Hamiltonian

$$\mathcal{H} = \epsilon(a^\dagger a + b^\dagger b) + g(a^\dagger b^\dagger + ba). \tag{3}$$

Introduce two new operators, c and d according to

$$\begin{aligned} a &= uc - vd^\dagger, \\ b &= ud + vc^\dagger. \end{aligned} \tag{4}$$

where u and v are arbitrary complex numbers. This is called a **Bogoliubov transformation**.

(a) Find which constraints are imposed on u and v when we require that c and d should also be Fermionic operators.

(b) Insert this transformation in the Hamiltonian (3) and choose u and v so as to kill the terms containing $c^\dagger d^\dagger$ and cd. This is how we diagonalize Hamiltonians of this form.