# Statistical Mechanics - Problem set 5

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Deadline: 10/12

1. **Landau theory for discontinuous transitions:** Consider a system described by a real order parameter $m$ with $Z_2$ symmetry ($m \to -m$). In class we discussed how to expand the Landau free energy close to the critical point. In order to respect the $Z_2$ symmetry, the expansion could contain only even terms. Hence, it would look like

$$f(m) = \frac{a}{2}m^2 + \frac{b}{4}m^4 + \frac{c}{6}m^6. \tag{1}$$

In class we stopped at the quartic term because we were assuming that $b > 0$. However, if $b < 0$ then we need to go one order further in order to get a thermodynamically stable theory (we hence assume that $c > 0$. If not, we have to keep going...). We also continue to have $a \sim T - T_c$, so that it changes sign at the critical point.

   (a) Assuming $b < 0$, show that if

$$a > a_0 := \frac{b^2}{4c}, \tag{2}$$

   then the only minimum is at $m = 0$. Conversely, if $a < a_0$ two new minima appear at $m \neq 0$. However, for $a$ just below $a_0$ these will be local minima. They therefore correspond to **metastable states**: they are not global minima, but there is nonetheless a barrier separating them from the true minimum, so that it would require some energy to remove the system from them.

   (b) Show that the solutions with $m \neq 0$ become global minima when

$$a < a_1 := \frac{3b^2}{16c}. \tag{3}$$

   This is the critical point for a discontinuous transition: when $a$ reaches $a_1$ the system will jump abruptly from $m = 0$ to the state with $m \neq 0$. This is fundamentally different from the situation we studied in class, where $m$ varied continuously.

   (c) For $a$ just below $a_1$, the configuration with $m = 0$ continues to be a local minimum. Show, however, that if $a < 0$, this local minimum is completely destroyed.

2. **Superfluid He4-He3 mixtures:** Superfluid helium-4 is described by a complex order parameter $\phi(\boldsymbol{x})$. If, however, the system contains helium-3 impurities, we can take them into account by introducing a real field $A(\boldsymbol{x})$. The Landau-Ginzburg free energy is given by

$$F[\phi(\boldsymbol{x}), A(\boldsymbol{x})] = \int d^d\boldsymbol{x}\left\{|\nabla\phi|^2 + \frac{a}{2}|\phi|^2 + \frac{b}{4}|\phi|^4 + \frac{c}{6}|\phi|^6 + \frac{A^2}{2\sigma^2} - \lambda|\phi|^2 A\right\}, \tag{4}$$

where $b, c > 0$. In this problem we will focus only on the equilibrium solution, so to simplify we can take assume both $\phi(\boldsymbol{x})$ and $A(\boldsymbol{x})$ are homogenous in space, leading simply to a Landau free energy of the form

$$f(\phi, A) = \frac{a}{2}|\phi|^2 + \frac{b}{4}|\phi|^4 + \frac{c}{6}|\phi|^6 + \frac{A^2}{2\sigma^2} - \lambda|\phi|^2 A. \tag{5}$$

(a) Find the value of $A^*$ which minimizes $f$. Then substitute $A^*$ in Eq. (5) to find an effective free energy $f_{\rm eff}(\phi)$ for the order parameter $\phi$ only.

(b) Show that this effective free energy may have both a continuous as well a discontinuous transition, depending on the value of $\sigma$. Determine the critical value $\sigma^*$ at which the transition changes from one type to the other. Tip: use what you learned in problem 1.

3. **The Higgs mechanism:** Consider a Landau-Ginzburg theory for a system that has two order parameters: a complex field $\phi(\boldsymbol{x})$ and a scalar field $A(\boldsymbol{x})$. The Landau-Ginzburg free is taken to be of the form

$$F[\phi(\boldsymbol{x}), A(\boldsymbol{x})] = \int \mathrm{d}^d\boldsymbol{x}\Big\{|\nabla\phi|^2 + \frac{a}{2}|\phi|^2 + \frac{b}{4}|\phi|^4 + (\nabla A)^2 + \lambda|\phi|^2A^2\Big\}, \tag{6}$$

where $\lambda > 0$, $b > 0$ and $a$ changes sign at the phase transition.

(a) Find the values of $\phi$ and $A$ which minimize $F$. Tip: this requires no fancy calculations. Just think a bit about how $A(\boldsymbol{x})$ affects the minimization.

(b) Expand the fields $\phi(\boldsymbol{x})$ and $A(\boldsymbol{x})$ around the equilibrium point, keeping only the terms which are at most quadratic in the fluctuations.

(c) Show that when $\phi(\boldsymbol{x})$ is in a broken symmetry phase, the field $A(\boldsymbol{x})$ acquires a mass term. This is the Higgs mechanism: one can make the field $A$ massive, by coupling it to another field ($\phi$) which is in a broken symmetry phase.