Thermodynamics of continuously measured quantum systems

Gabriel T. Landi
Universidade de São Paulo, Brazil

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www.fmt.if.usp.br/~gtlandi

In collaboration with

- Mauro Paternostro, Alessio Belenchia, Luca Mancino (Belfast).
- Massimiliano Rossi, Albert Schliesser (Copenhagen).

Entropy Production in Continuously Measured Quantum Systems

Alessio Belenchia,[1] Luca Mancino,[1] Gabriel T. Landi,[2] and Mauro Paternostro[1]

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Thermodynamics of quantum measurements: Insights from classical collisional models

Gabriel T. Landi,[1,*] Mauro Paternostro,[2] and Alessio Belenchia[2]

In preparation.

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Editors' Suggestion

Experimental Assessment of Entropy Production in a Continuously Measured Mechanical Resonator

Massimiliano Rossi,[1,2] Luca Mancino,[3] Gabriel T. Landi,[4] Mauro Paternostro,[3] Albert Schliesser,[1,2] and Alessio Belenchia[3,*]

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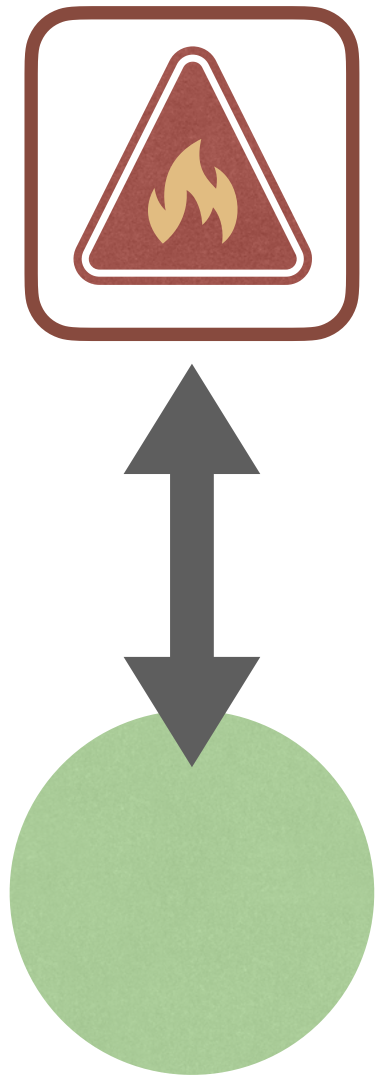
⚛️ 2nd law at the quantum level

- Consider a physical system S with an arbitrary initial state ρ_S and interacting with a bath E prepared in a thermal state $\rho_E = e^{-\beta H_E}/Z_E$ via an arbitrary unitary U:

$$\rho'_{SE} = U(\rho_S \otimes \rho_E)U^\dagger$$

- This describes a very broad class of processes!
  - from an atom interacting with the electromagnetic vacuum…
  - …to a red-hot sword being dipped in a bucket of water.
- The unitary may be insanely complicated.
  - But the map will still be of this form.



GTL and M. Paternostro, "Irreversible entropy production, from quantum to classical", arXiv:2009.07668

- Let $S(\rho_S) = -\,\mathrm{tr}(\rho_S \ln \rho_S)$ denote the entropy of the system.
- It was shown by Esposito and Lindenberg that

$$\Sigma := \Delta S_S - \beta Q_E \geqslant 0$$

where $Q_E = \langle H_E \rangle' - \langle H_E \rangle$ is the heat that entered the bath.

- This implies that the changes in entropy of the system are not independent of the heat flux to the bath.
- Σ is called the **entropy production**.
- This is the 2nd law in a fully quantum formulation.
  - ✓ It can be extended to multiple baths.
  - ✓ And reproduces classical results in the appropriate limits.

Σ can be expressed as a fully information-theoretic quantity:

$$\Sigma = I'(S:E) + S(\rho_E' \,||\, \rho_E)$$

where

$$I'(S:E) = S(\rho_S') + S(\rho_E') - S(\rho_{SE}')$$

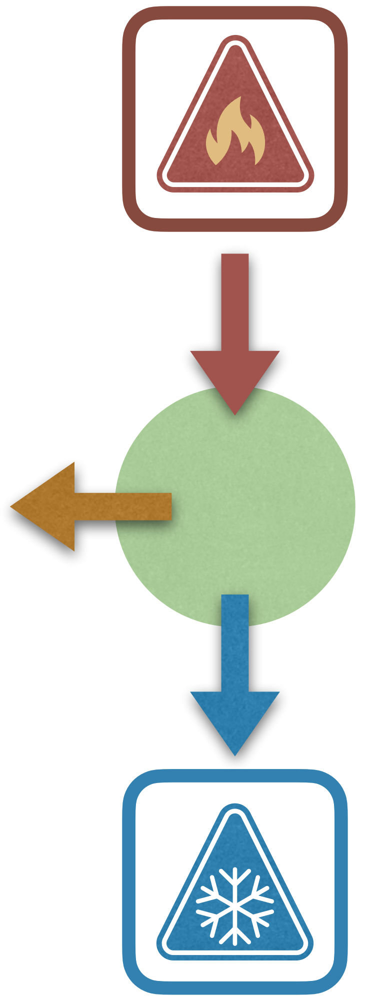
$$S(\rho_E' \,||\, \rho_E) = \mathrm{tr}\left(\rho_E' \ln \rho_E' - \rho_E' \ln \rho_E\right)$$

M. Esposito, K. Lindenberg, C. Van den Broeck, "Entropy production as correlation between system and reservoir". New Journal of Physics, **12**, 013013 (2010).

- Σ measures how irreversible a process is.
- Example:
    - The efficiency of a heat engine can be written as

$$\eta = \eta_C - \frac{T_c \Sigma}{Q_h}, \qquad \eta_C = 1 - T_c/T_h$$

    - Since $\Sigma \geqslant 0$ (2nd law), it follows that $\eta \leqslant \eta_C$.
- Example:
    - Suppose U is generated by an infinitesimal quench $H_S(0) \to H_S(1) = H_S(0) + \delta H_S$.
    - Then work $\langle W \rangle \sim \delta H$ but $\Sigma \sim \delta H_S^2$.



M. Scandi, H.J.D. Miller, J. Anders and M. Perarnau-Llobet, "Quantum work statistics close to equilibrium", Phys. Rev. Research **2**, 023377 (2020).

✳️ Production/flux in non-equilibrium settings

- The idea of entropy production, as a gauge of irreversibility can also be extended beyond thermal environments.

- The map continues to have the form:

$$\rho'_{SE} = U(\rho_S \otimes \rho_E)U^\dagger$$

  but with arbitrary ρ_E.

- The entropy production is still defined as

$$\Sigma = I'(S:E) + S(\rho'_E||\rho_E)$$

- This can always be written as

$$\Sigma = \Delta S_S + \Phi$$

  where

$$\Phi = \mathrm{tr}_E\Big\{(\rho_E - \rho'_E)\ln\rho_E\Big\}$$

  is called the **entropy flux**.

- Φ depends only on E. It measures the change in the "thermodynamic potential" $\ln\rho_E$ of the environment.

- For thermal baths, Φ coincides with the heat flux.

🛂 Conditional entropy production

Conditional entropy production

- Part of the irreversibility stems from our ignorance about the environment.

- Suppose we measure E after it interacted with S.

$$\rho'_{SE} \to \rho'_{SE|k} = (1 \otimes M_k)\rho'_{SE}(1 \otimes M_k^\dagger)$$

$$p_k = \mathrm{tr}_E\left(M_k^\dagger M_k \rho'_E\right)$$

$\{M_k\}$ = generalized measurement operators acting on E:

This is a conditional state. It is the state of SE, conditioned on the measurement outcome being k.

- What is the entropy production and flux, conditioned on these outcomes? That is, we are looking for something like

$$\Sigma_k = S(\rho'_{S|k}) - S(\rho_S) + \Phi_k$$

- Or, focusing on the average over all outcomes,

$$\Sigma_c = \sum_k p_k S(\rho'_{S|k}) - S(\rho_S) + \Phi_c$$

How to define Σ_c, Φ_c?

- A natural generalization of $\Phi = \mathrm{tr}_E\Big\{(\rho_E - \rho_E')\ln\rho_E\Big\}$ is

$$\Phi_k = \mathrm{tr}_E\big\{(\rho_E - \rho_{E|k}')\ln\rho_E\big\}$$

- Averaging over p_k yields

$$\Phi_c = \sum_k p_k \Phi_k = \Phi = \mathrm{tr}\big\{(\rho_E - \tilde{\rho}_E)\ln\rho_E\big\}, \qquad \tilde{\rho}_E = \sum_k p_k \rho_{E|k}'$$

- If the measurement is non-disturbing then $\tilde{\rho}_E = \rho_E'$.
    - In this case the conditional and unconditional fluxes coincide.
    - This makes sense: if this is to be a flux, then it shouldn't depend on the subjective information one has about the measurement.
        - It can still depend on a possible disturbance caused by the measurement. But we are going to assume this is not the case.

- The unconditional and conditional $\Sigma's$ are thus

$$\Sigma = S(\rho_S') - S(\rho_S) + \Phi$$

$$\Sigma_c = \sum_k p_k S(\rho_{S|k}') - S(\rho_S) + \Phi$$

- Whence,

$$\Sigma_c = \Sigma - \chi_M(\rho_S')$$

where

$$\chi_M(\rho_S') = S(\rho_S') - \sum_k p_k S(\rho_{S|k}') = \sum_k p_k S(\rho_{S|k}' || \rho_S')$$

is the Holevo quantity ✅.

- One may show that

$$0 \leqslant \Sigma_c \leqslant \Sigma$$

- Thus, the conditional entropy production still satisfies a 2nd law ($\Sigma_c \geqslant 0$).

- But it is also smaller than the unconditional one:
  - Conditioning makes the process more reversible.

K. Funo, Y. Watanabe and M. Ueda, "Integral quantum fluctuation theorems under measurement and feedback control". PRE, **88**, 052121 (2013).

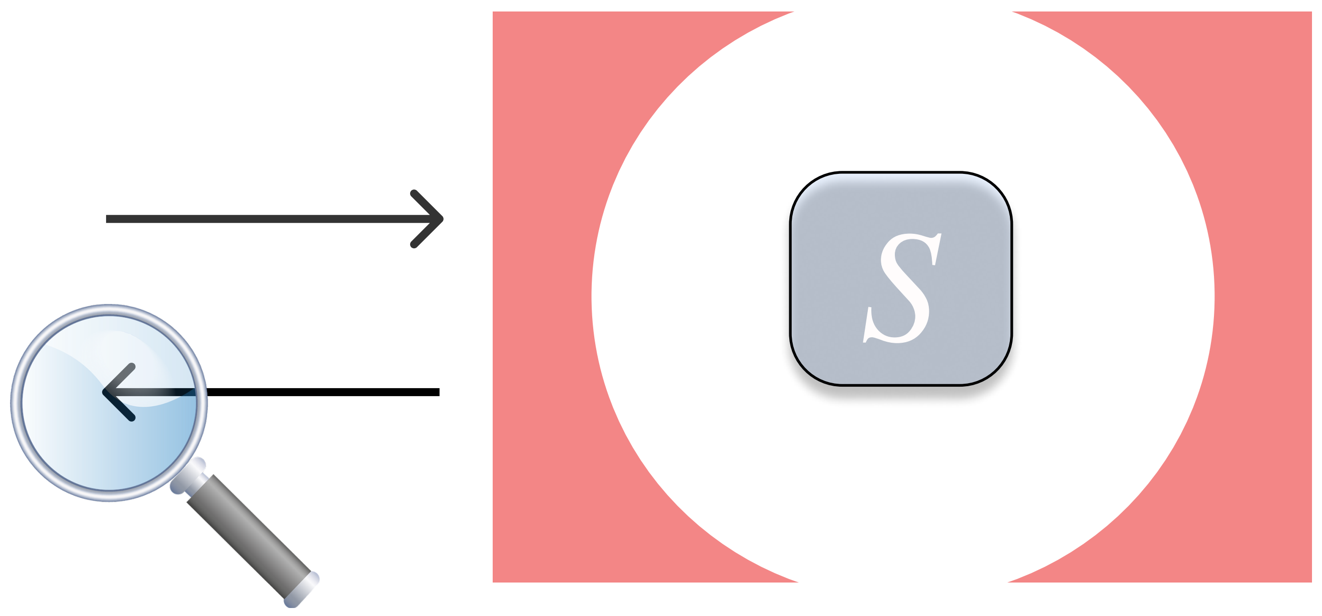
GTL and M. Paternostro, "Irreversible entropy production, from quantum to classical", arXiv:2009.07668

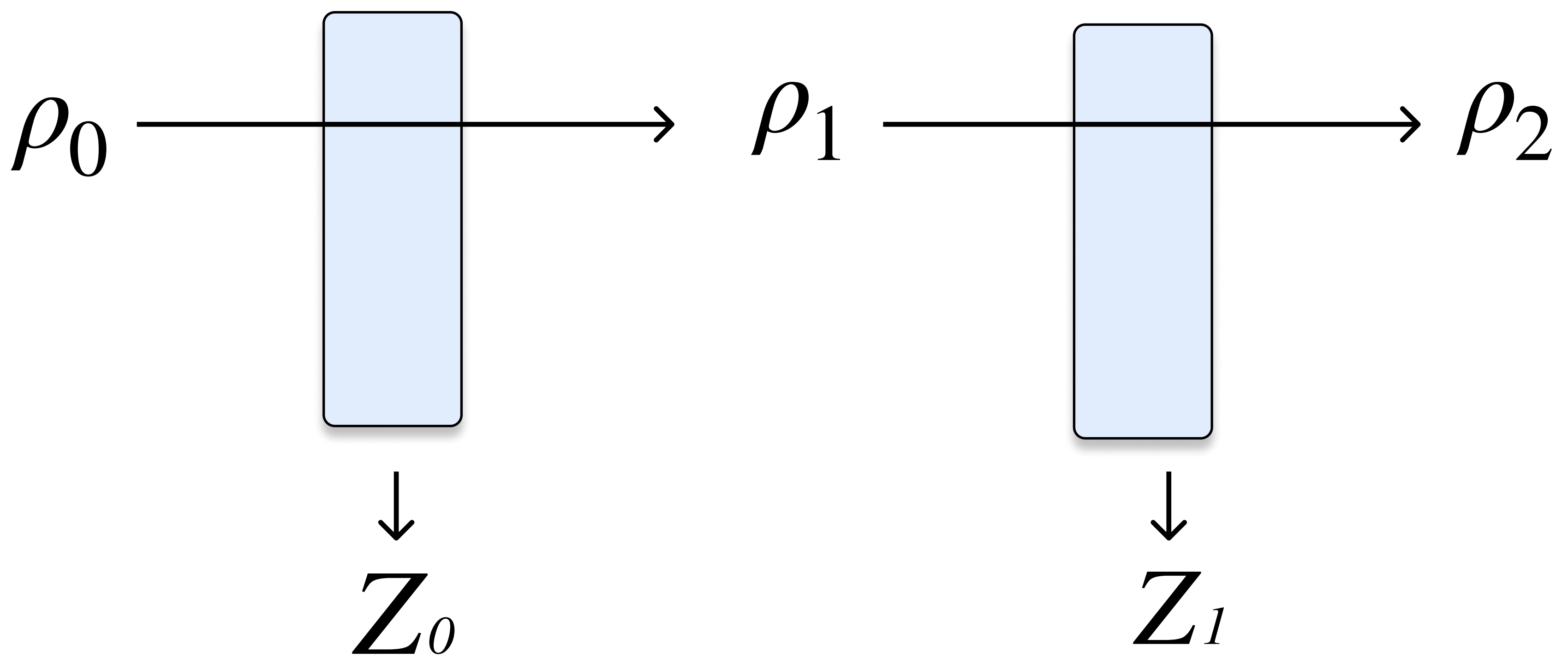
M. Naghiloo, J. J. Alonso, A. Romito, E. Lutz, K. Murch, "Information Gain and Loss for a Quantum Maxwell's Demon". PRL **121**, 030604 (2018).

🔍 Continuous weak measurements

Continuous weak measurements

- What about systems that are continuously monitored by a weak probe?
- Things become more complicated because now we have the entire **measurement record** to take into account.
  - For instance, there will be both integral and differential information gains.





H. M. Wiseman and G. J. Milburn, "Quantum Measurement and Control".
K. Jacobs, "Quantum Measurement Theory".

Gaussian continuous weak measurements

- The theory of continuous measurements is further developed, and can go much deeper, in the case of continuous variables undergoing Gaussian-preserving dynamics.

- Let $x = (q_1, p_1, q_2, p_2, \ldots)$ denote the vector of quadrature operators. Gaussian systems are fully characterized by their 2 first moments:

  - the average $\bar{x} = \langle x \rangle$

  - and the covariance matrix (CM) $\sigma_{ij} = \frac{1}{2}\langle \{x_i, x_j\} \rangle - \langle x_i \rangle \langle x_j \rangle$.

- We must track both the conditional and unconditional dynamics.

  - Unconditional means we monitor (there is still backaction) but we don't care about the results. Described by a Lindblad MEq.

  - Conditional dynamics is stochastic because we condition on random outcomes. Described by a stochastic MEq.

A. Serafini, "Quantum Continuous Variables: A Primer of Theoretical Method".

M. G. Genoni, L. Lami, and A. Serafini, "Conditional and unconditional Gaussian quantum dynamics", Contemp. Phys. **57**, 331 (2016).

- Unconditional variables evolve as in a Lindblad master equation:

$$\frac{d\bar{x}_u}{dt} = A\bar{x}_u + b$$

where A, b depend on both unitary and dissipative dynamics.

- Similarly, the CM evolves according to the Lyapunov equation:

$$\frac{d\sigma_u}{dt} = A\sigma_u + \sigma_u A^T + D$$

where D is called the diffusion matrix.

- The continuous measurement will cause the mean $\bar{x}_c$ to evolve stochastically according to the Langevin equation:

$$\frac{d\bar{x}_c}{dt} = (A\bar{x}_c + b) + (\sigma_c C^\mathsf{T} + \Gamma^\mathsf{T})\xi(t)$$

where C, Γ are matrices and $\xi(t)$ is a vector of white noises.

- The CM, on the other hand, evolves deterministically:

$$\frac{d\sigma_c}{dt} = A\sigma_c + \sigma_c A^\mathsf{T} + D - \chi(\sigma_c)$$

where

$$\chi(\sigma) = (\sigma_c C^\mathsf{T} + \Gamma^\mathsf{T})(C\sigma + \Gamma) \geqslant 0$$

describes the information gained due to the measurement.

M. G. Genoni, L. Lami, and A. Serafini, "Conditional and unconditional Gaussian quantum dynamics", Contemp. Phys. **57**, 331 (2016).

🔁 Thermodynamics of Gaussian CMs

- In the case of continuous measurements, the relevant quantity is the entropy production **rate**.

- We formulate the thermodynamics of this model using a semi-classical representation in terms of the Wigner function $W(x)$ (standard approach does not work).

  - The Wigner function, conditioned on a given outcome for the average, is $W_c(x|\bar{x})$.

  - The variable $\bar{x}$ is classical, with probability distribution $p(\bar{x})$.

  - The conditional and unconditional Wigner functions are thus associated by a Kalman filter:

$$W_u(x) = \int W_c(x|\bar{x})p(\bar{x})d\bar{x}$$

A. Belenchia, L. Mancino,GTL and M. Paternostro, “Entropy Production in Continuously Measured Quantum Systems”, arXiv:1908.09382. Accepted in npj Quantum Information.

- As an alternative representation of entropy, we can use

$$S_u = -\int W_u(x) \ln W_u(x) dx$$

and

$$S_c = -\int p(\bar{x}) d\bar{x} \int W_c(x|\bar{x}) \ln W_c(x|\bar{x}) dx$$

- Their difference represents the net amount of information acquired by the measurement record:

$$I = S_u - S_c \geqslant 0$$

- This is the phase-space analog of the Holevo quantity. Exactly the same idea ✅.

$$\left(\chi_M(\rho'_S) = S(\rho'_S) - \sum_k p_k S(\rho'_{S|k}) \right)$$

G. Adesso, D. Girolami, A. Serafini, "Measuring gaussian quantum information and correlations using the Rényi entropy of order 2". PRL **109**, 190502 (2012).

🅿️ Unconditional production/flux

- The unconditional Wigner function evolves according to a Fokker-Planck equation:

$$\frac{\partial W}{\partial t} = \mathrm{div}\left[J + J_{\mathrm{sto}}\right]$$

where

$$J = (Ax + b)W - \frac{D}{2}\nabla W$$

is a quasi-probability current.

- The entropy production and flux rates are

$$\Pi_u = 2\int \frac{dx}{W_u}\, J^T D^{-1} J \geqslant 0$$

$$\Phi_u = -2\int J^T D^{-1} A\, dx$$

- The stochastic MEq is translated into a stochastic Fokker-Planck equation:

$$\frac{\partial W_c}{\partial t} = \mathrm{div}\left[J + J_{\mathrm{sto}}\right]$$

where

$$J_{\mathrm{sto}} = W_c(\sigma_c C^T + \Gamma^T)\xi(t)$$

- One can show that the flux does not change:

$$\Phi_c = \Phi_u$$

as intuitively expected.

J. P. Santos, GTL, M. Paternostro, "The Wigner entropy production rate", PRL **118**, 220601 (2017).

- Hence, as before, we will have

$$\Pi_u = \dot{S}_u + \Phi_u \qquad \therefore \qquad \Pi_c = \Pi_u - \dot{I}$$
$$\Pi_c = \dot{S}_c + \Phi_u$$

- In particular, the net rate of information gain can be shown to be

$$\dot{I} = \frac{1}{2}\mathrm{tr}\left[D(\sigma_c^{-1} - \sigma_u^{-1})\right] - \frac{1}{2}\mathrm{tr}\left[\chi(\sigma_c)\sigma_c^{-1}\right] := \dot{L} - \dot{G}$$

  ✓ The 1st term is the information loss rate due to the dissipation ($\propto D$).

  ✓ The 2nd term is the information gain rate, due to the update matrix $\chi(\sigma_c)$

- In the steady-state $\dot{I} = 0$. But this does not mean we are no longer acquiring information.
  - What it means is that $\dot{G} = \dot{L}$: the information acquired is balanced by the information dissipated.

Informational steady-state

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🇩🇰 Copenhagen setup

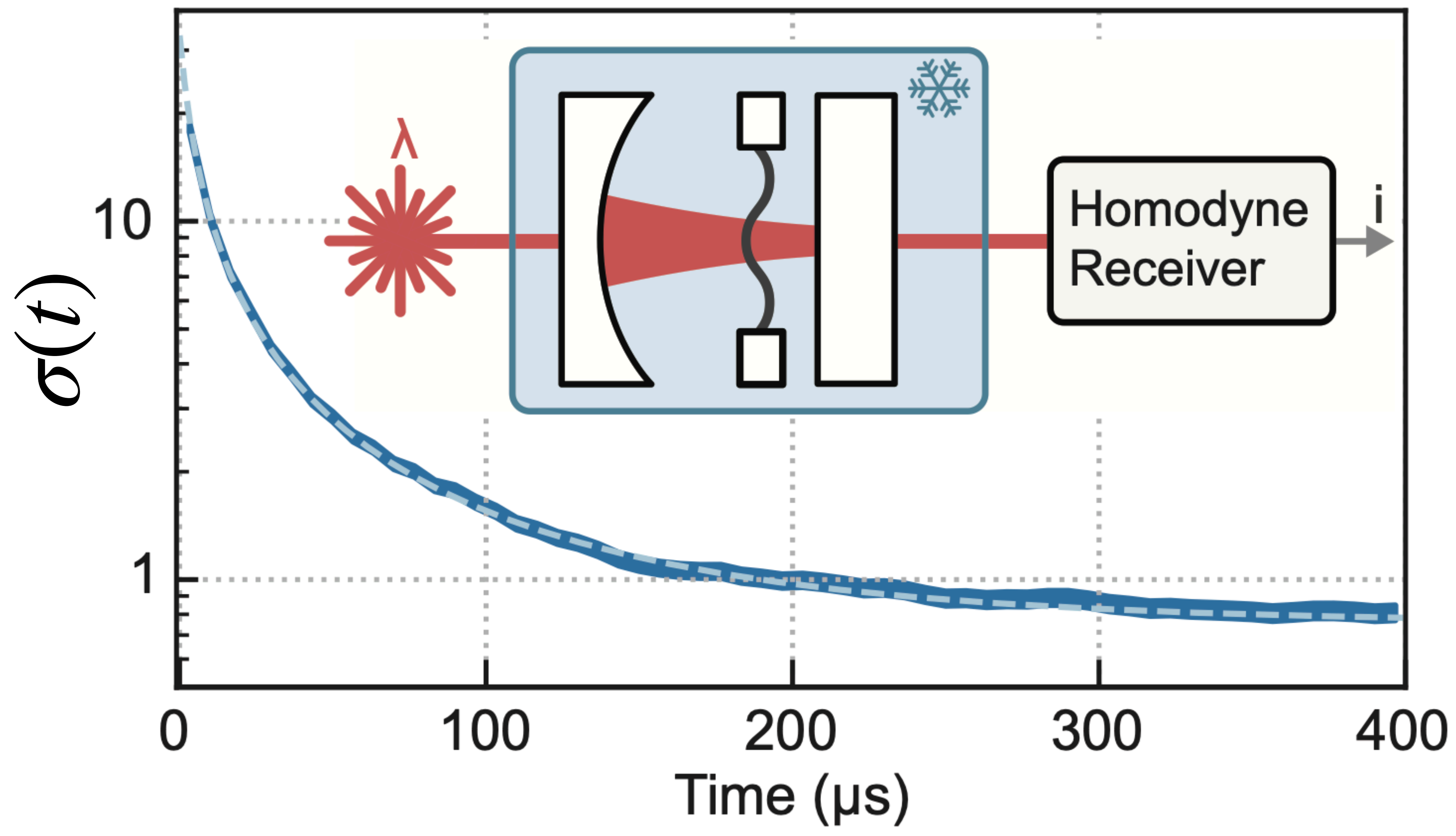
- Optomechanical system continuously monitored by an optical field.
- Competition: Thermal bath vs. Measurement.
- Quadratures of the mechanical mode: $x = (q, p)$
- CM $\sigma \propto \mathbb{1}$
- Unconditional dynamics tends to $\bar{x}_u = 0$

$$\sigma_u = \bar{n} + 1/2 + \Gamma_{qba}/\Gamma_m$$

- Conditional dynamics evolves instead to

$$\frac{dx}{dt} = -\frac{\Gamma_m}{2}x + \sqrt{4\eta\Gamma_{\mathtt{qba}}}\sigma_c(t)\xi(t)$$

$$\frac{d\sigma_c}{dt} = \Gamma_m(\sigma_u - \sigma_c) - 4\eta\Gamma_{\mathtt{qba}}\sigma_c^2$$



Informational steady-state:

Conditional dynamics relaxes to a colder state, $\sigma_c < \sigma_u$, which can only be maintained by continuously monitoring S.

$$H = \omega a^\dagger a + \left(\frac{p}{2m} + \frac{1}{2}\omega^2 x^2\right) + g a^\dagger a x$$

Production and flux at the trajectory level

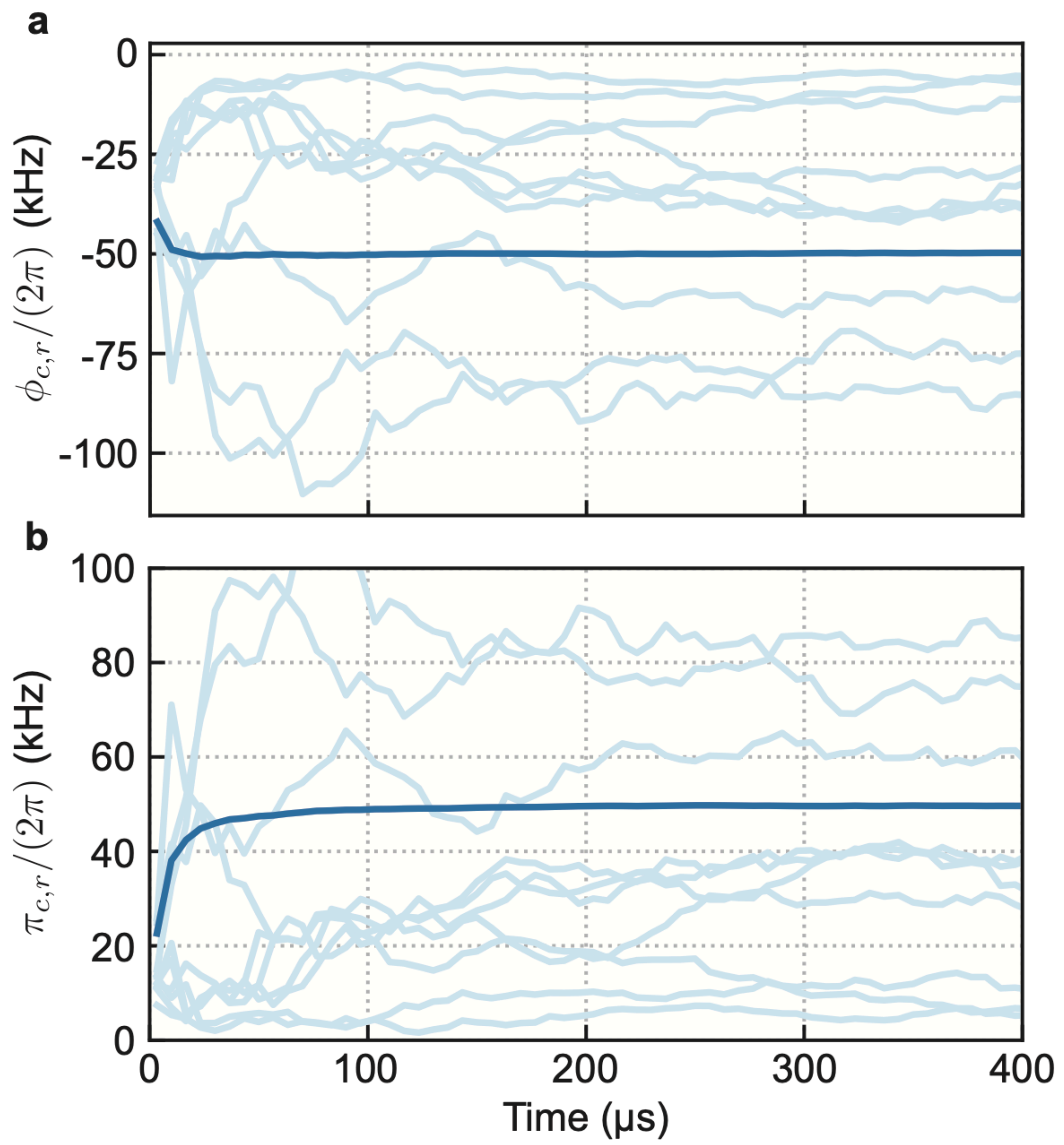


FIG. 2. **Stochastic entropy flux and production rates.** **a,** The stochastic entropy flux rates (light blue) for a sample of 10 trajectories. The dark blue line is the ensemble average over all the trajectories. **b,** The stochastic entropy production rates (light blue) and the ensemble average (dark blue), for the same sample of trajectories.

Information gain/loss rates characterizing the information steady-state

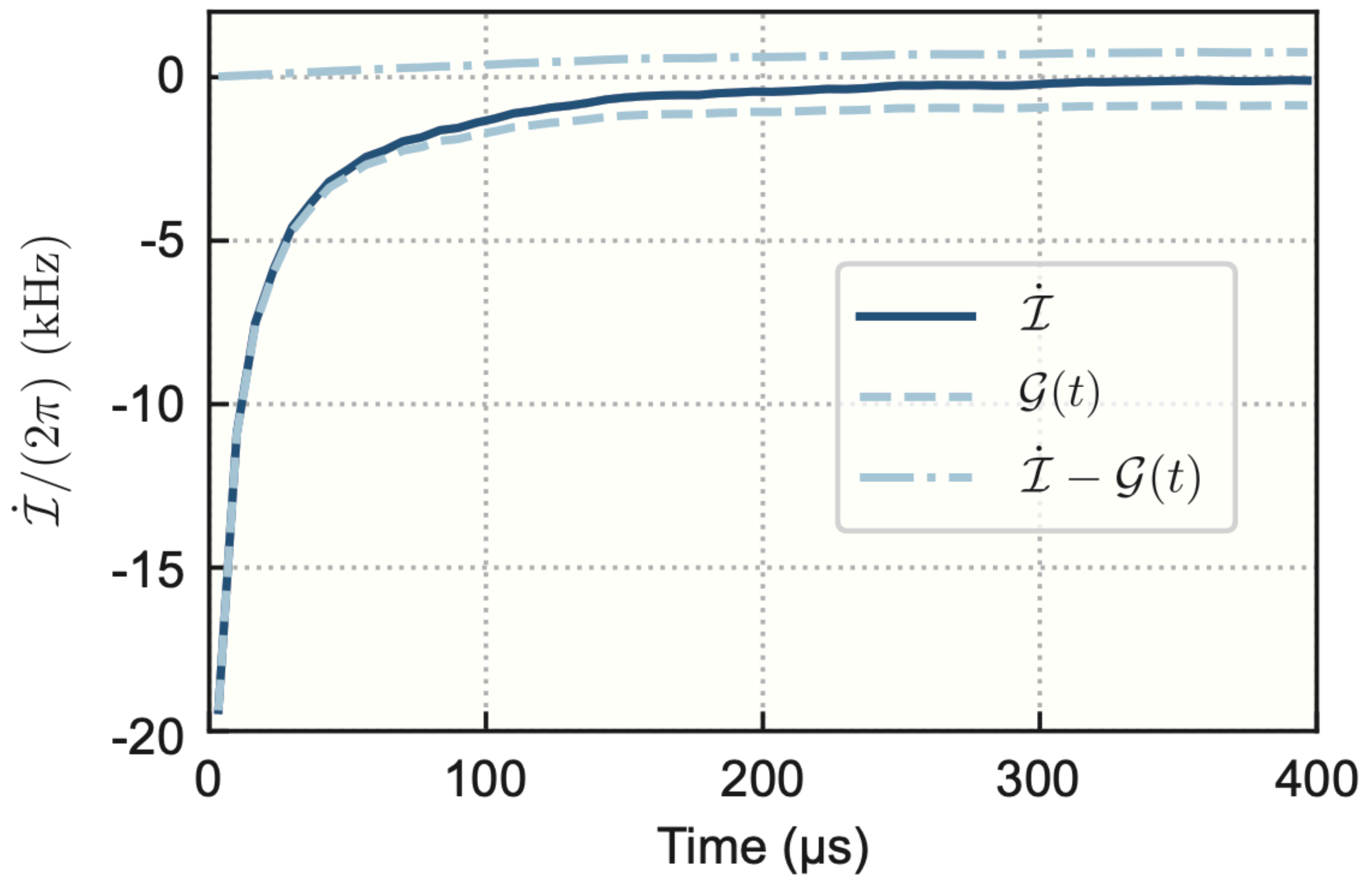


FIG. 3. **Informational contribution to the entropy production rate.** We obtain the informational contribution (dark blue) from the entropy production. The dashed (dot-dashed) line is the differential gain of information due to the measurement (loss of information due to noise input by the phonon bath).

Conclusions

- Knowing something about the bath makes the process less irreversible.
- The conditional entropy production quantifies this effect.
- But quantifying this for continuously monitored quantum systems is not trivial.
  - We put forth a framework for GCV systems.
    - Rich and clear physical interpretation.
  - We also provide an experimental assessment of the entropy production at the level of stochastic trajectories in a quantum optomechanical system.

Thank you! 🙃

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