Thermodynamic analysis of quantum error-correcting engines

Gabriel T. Landi,[1,*] André L. Fonseca de Oliveira,[2] and Efrain Buksman[2]
[1]*Instituto de Física da Universidade de São Paulo, 05314-970 São Paulo, Brazil*
[2]*Facultad de Ingeniería, Universidad ORT Uruguay, CP 11100 Montevideo, Uruguay*

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Quantum error-correcting codes can be cast in a way which is strikingly similar to a quantum heat engine undergoing an Otto cycle. In this paper, we strengthen this connection further by carrying out a complete assessment of the thermodynamic properties of four-stroke operator-based error-correcting codes. This includes an expression for the entropy production in the cycle which, as we show, contains clear contributions stemming from the different sources of irreversibility. To illustrate our results, we study a classical three-qubit error-correcting code, well suited for incoherent states, and the nine-qubit Shor code capable of handling fully quantum states. We show that the work cost associated with the correction gate is directly associated with the heat introduced by the error. Moreover, the work cost associated with encoding and decoding quantum information is always positive, a fact which is related to the intrinsic irreversibility introduced by the noise. Finally, we find that correcting the coherent (and thus genuinely quantum) part of a quantum state introduces substantial modifications related to the Hadamard gates required to encode and decode coherences.

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I. INTRODUCTION

Quantum error-correcting codes (QECCs) protect qubits from detrimental noise by redundantly storing quantum states in multiple parties [1]. The basic idea is illustrated in Fig. 1. An environment induces noise in a system S, which is modeled as a quantum channel $\mathcal{E}_H$, as in Fig. 1(a). In order to protect it, the state of the system is encoded into a larger Hilbert space by introducing additional ancillas. Both system and ancillas are now susceptible to the noise process. But by applying appropriate correction measures, one may mitigate this noise at the expense of making the final state of the ancillas more mixed [Fig. 1(b)].

The connection between QECCs and thermodynamics has been discussed for quite some time in connection with Landauer's erasure and Maxwell's Demon [2,3]. However, an interesting connection is that with quantum heat engines (QHEs) [4,5]. This becomes accurate in the case of operator error correction [6–8], where no syndrome measurements are required. The diagram in Fig. 1 is then seen to be entirely analogous to a quantum heat engine undergoing an Otto cycle [9,10]: The "working fluid" is composed of both system and ancillas. The encoding, decoding, and correction steps are the unitary strokes, involving the expenditure of work without any heat flow. The noise term represents the action of the hot bath. And, finally, the recycling step where the states of the ancillas are reset represents the cold bath.

In view of this striking similarity, one is naturally led to ask how far this connection can be pushed. Of course, in the end, the goal of a QECC is entirely different from that of a QHE. Efficiency, for instance, has nothing to do with work extraction, but with the ability of the code to correct the error. Notwithstanding these fundamental differences, an analysis of a QECC from a thermodynamic perspective is still illuminating, as it allows one to address the roles of heat and work in the error-correcting process. Particularly interesting is the question of what the work cost is for encoding and decoding quantum information, as compared to the cost for applying an error correction. For instance, is it possible to successfully apply an error correction and still extract useful work from the machine? Or does the success of the QECC necessarily involve the expenditure of work by an external agent?

With these motivations in mind, in this paper we put forth a complete thermodynamic characterization of QECCs implemented using the operator error-correction scheme [6,7]. We begin by considering the general thermodynamic properties, including a reformulation of the first and second laws for the specific QECC scenario. Next we apply these results to two representative examples. The first is a three-qubit classical error correction, capable of correcting incoherent states. The second is the fully quantum nine-qubit Shor code, which can simultaneously handle both incoherent and coherent states.

II. FORMAL FRAMEWORK

In this section, we provide a general characterization of the thermodynamic properties of the QECC in Fig. 1. We begin by describing the basic strokes of the cycle and then move on to characterize it in terms of the first and second laws of thermodynamics.

A. Description of the cycle

We assume the main system S is a qubit with computational basis $|0\rangle, |1\rangle$ and Hamiltonian $H_S = \frac{\Omega}{2}(1 - \sigma_z^S)$ (so that $|0\rangle$ is the ground state). The code involves coupling the system

*gtlandi@if.usp.br

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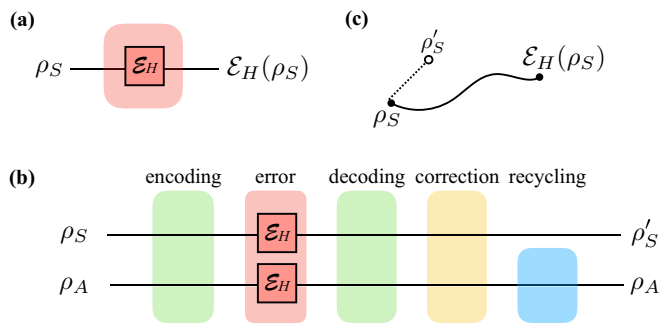

FIG. 1. Typical error-correcting scenario. (a) A state ρ_S is susceptible to error, described by a quantum channel $\mathcal{E}_H(\rho_S)$. (b) In order to protect it, ρ_S is first encoded into ancillas A. After both undergo individual errors $\mathcal{E}_H$, the state of the system is decoded from $S+A$ and a set of correction measures is applied, leading to a final state ρ'_S for the system. This procedure makes the ancillas dirty, which must then be recycled if they are to be used again. (c) The code is considered successful (at the ensemble level) whenever it mitigates the role of the noise, which means $D(\rho'_S, \rho_S) < D(\mathcal{E}_H(\rho_S), \rho_S)$, where D is any distance measure. This, of course, will be the case only if the effect of $\mathcal{E}_H$ is sufficiently small.

with a set of ancillas, which we shall henceforth assume to be identical, with Hamiltonian $H_{A_i} = \frac{\omega}{2}(1-\sigma_z^{A_i})$. The ancillas are always prepared in the ground state $|0\rangle$, so that the global initial state of N ancillas is $\rho_A = |0\rangle\langle 0|^{\otimes N}$. Below, when convenient, we shall assume for simplicity that $\omega = \Omega$.

In this paper, we will consider four-stroke codes, each of which we now explain in detail. The first stroke is the *encoding stroke*, where the system density matrix ρ_S is encoded in the ancillas by means of a unitary U_e,

$$\rho_{SA}^{(1)} = U_e(\rho_S \otimes \rho_A)U_e^\dagger. \tag{1}$$

The second stroke is the *error (noise) stroke*, where both S and A are subject to local noise channels. In order to highlight the correction with thermodynamics, we consider the noise generated by the generalized amplitude damping (GAD) channel,

$$\mathcal{E}_H(\rho) = \sum_{k=1}^{4} M_k \rho M_k^\dagger, \tag{2}$$

where

$$M_1 = \sqrt{1-f}\begin{pmatrix}1 & 0\\ 0 & \sqrt{1-\gamma}\end{pmatrix},\quad M_2 = \sqrt{1-f}\begin{pmatrix}0 & \sqrt{\gamma}\\ 0 & 0\end{pmatrix},$$
$$M_3 = \sqrt{f}\begin{pmatrix}\sqrt{1-\gamma} & 0\\ 0 & 1\end{pmatrix}, M_4 = \sqrt{f}\begin{pmatrix}0 & 0\\ \sqrt{\gamma} & 0\end{pmatrix}. \tag{3}$$

Here, $\gamma \in [0,1]$ is the coupling strength and f is the excited-state probability (Fermi-Dirac distribution). If $f=0$, the map will target the ground state $|0\rangle$. Since S and A have different frequencies, we will use the notation $f_x = (e^{\beta_H x}+1)^{-1}$, with $x = \Omega, \omega$ and β_H being the temperature of the hot bath. Error correction is mostly successful when the noise strength $\gamma \ll 1$, which we shall assume throughout this paper. Moreover, following customary treatments of error correction, all results for specific codes will be presented in terms of a power series, only to leading order in γ. The state after the second stroke will be

$$\rho_{SA}^{(2)} = \mathcal{E}_H^S \otimes \mathcal{E}_H^{\otimes N}(\rho_{SA}^{(1)}). \tag{4}$$

The third stroke is the *decoding and correction* operation. This will again be described by a unitary U_{dc}, which, in general, cannot be split as the product of two unitaries for decoding and correction. The state after the third stroke will be

$$\rho_{SA}^{(3)} = U_{dc}\rho_{SA}^{(2)}U_{dc}^\dagger. \tag{5}$$

Finally, the fourth stroke is the *recycling* stroke, where the ancillas interact with a cold bath and nothing is done to the system. This stroke can also be viewed as the action of a GAD (2), but with $\gamma = 1$ and $f = 0$. However, this is not necessary since its effect is simply to reset the state of the ancillas. Hence, after the fourth and final stroke, the global state will be

$$\rho_{SA}^{(4)} = \rho_S^{(3)} \otimes \rho_A, \tag{6}$$

where $\rho_S^{(3)} = \mathrm{tr}_A\, \rho_{SA}^{(3)}$ is the state of S after the fourth stroke and ρ_A was the initial state of the ancillas. We call attention here to the fact that, in general, the final state of the system after a cycle will be different from the initial cycle. This, of course, is precisely the goal of a QECC. However, it is somewhat unconventional from a thermodynamic perspective. In the language of thermodynamics, this would actually correspond to a transient operation of the engine. If the process was repeated multiple times, the system would reach a steady state, where the engine would no longer be able to correct anything.

B. Error-correcting efficiency

For conciseness, we shall denote $\rho'_S = \rho_S^{(3)} = \rho_S^{(4)}$ as the final state of the system after one cycle. Thus, from a global perspective, the input state of the engine is $\rho_S \otimes \rho_A$ and the output state is $\rho'_S \otimes \rho_A$. We therefore see that, in general, the engine's operation is not cyclic (i.e., it has not reached a limit cycle). This, actually, is precisely what quantifies the efficiency of the error-correcting code, as the goal of the engine is to have ρ'_S as close as possible to ρ_S.

Motivated by this, one can define the efficiency of the QECC as follows. Let $D(\rho,\sigma)$ denote any proper distance measure between quantum states [11,12]. To address the success of a QECC, one must compare ρ'_S with the state $\mathcal{E}_H(\rho_S)$, which one would obtain if only the error map $\mathcal{E}_H$ were to be applied to the state. A QECC can be declared successful (at the ensemble level) if

$$D(\rho'_S, \rho_S) < D(\mathcal{E}_H(\rho_S), \rho_S), \tag{7}$$

since this implies that the effect of the noise was at least partially mitigated by the code. Hence, a proper measure of the efficiency of a QECC could be, for instance,

$$\eta_{\mathrm{QECC}} = 1 - \frac{D(\rho'_S, \rho_S)}{D(\mathcal{E}_H(\rho_S), \rho_S)}. \tag{8}$$

This quantity is 1 when the correction is perfect, zero when the correction has no effect, and negative when the code actually makes things worse. It resembles the thermodynamic efficiency, but is purely information theoretic. Below we will

not need this specific form of the QECC efficiency in order to construct the cycle's thermodynamic. We have presented it here simply to emphasize that the QECC efficiency is, in general, not at all related to any thermodynamic efficiency.

C. First law of thermodynamics

The operations described by the four strokes in Eqs. (1)–(4) are essentially implementing an Otto cycle [9,10]. Strokes 1 and 3 are unitary, involving the possible expenditure of work, but without any exchange of heat. Similarly, strokes 2 and 4 are purely dissipative, involving only the exchange of heat and no work. The expressions for the heat and work in each stroke are thus easily calculated as the changes in energy in each stroke, $W_e = \Delta H_{10}$, $Q_H = \Delta H_{21}$, $W_{dc} = \Delta H_{32}$, and $Q_C = \Delta H_{43}$, where $H = H_S + H_A$ is the total Hamiltonian and $\Delta H_{i,i-1} = \mathrm{tr}\,\{H(\rho_{SA}^{(i)} - \rho_{SA}^{(i-1)})\}$ is the total change in energy of each stroke.

The decoding and correction stroke (5) is described by a unitary U_{dc}, which can be decomposed as a riffling $U_{dc} = U_d^{(1)} U_c^{(1)} U_d^{(2)} U_c^{(2)} \ldots$, where $U_d^{(i)}$ and $U_c^{(i)}$ represent decoding and correcting steps, respectively. Based on this, the work W_{dc} can also be split as $W_{dc} = W_d + W_c$, giving the individual contributions from each part of the code.

The ancillas are reset after each stroke, but the system is not. Hence, as a consequence, the first law of thermodynamics reads

$$\Delta U_S = W_e + Q_H + W_{dc} + Q_C, \tag{9}$$

where $\Delta U_S = \mathrm{tr}\,\{H_S(\rho_S' - \rho_S)\}$ is the change in energy of the system only. Since the total Hamiltonian is split as $H = H_S + H_A$, the same may also be done for all heat and work contributions. Thus, we may also write the first law as

$$\Delta U_S = \left(W_e^S + Q_H^S + W_{dc}^S\right) + \left(W_e^A + Q_H^A + W_{dc}^A + Q_C^A\right), \tag{10}$$

where we used the fact that the heat to the cold bath only has an ancilla part. But since the operations are all local and since the state of the ancillas is reset, it follows that the last term must be identically zero. Hence, the first law can be written solely as a system-related quantity,

$$\Delta U_S = W_e^S + Q_H^S + W_{dc}^S. \tag{11}$$

D. Second law for the noise stroke

One can also write the second law of thermodynamics for the QECC. The heating stroke 2 involves a standard finite-temperature amplitude damping, for which the expression for the entropy production is very well established [13–17] and reads

$$\Sigma_H = \Delta S_{21} - \beta_H Q_H \geqslant 0, \tag{12}$$

where $\Delta S_{21} = S(\rho_{SA}^{(2)}) - S(\rho_{SA}^{(1)})$ is the change in von Neumann entropy $[S(\rho) = -\,\mathrm{tr}(\rho \ln \rho)]$ in stroke 2. The positivity of Σ_H is a direct consequence of the data-processing inequality [18].

This expression can be manipulated so as to better highlight the physical origins of the irreversibility associated with the QECC cycle. Since stroke 3 is unitary, it follows that $S(\rho_{SA}^{(2)}) = S(\rho_{SA}^{(3)})$. Moreover, we can write

$$S\left(\rho_{SA}^{(3)}\right) = S(\rho_S') + S\left(\rho_A^{(3)}\right) - \mathcal{I}^{(3)}(S:A),$$

where $\mathcal{I}^{(3)}(S:A)$ is the mutual information between the system and ancilla in the state $\rho_{SA}^{(3)}$. Similarly, since the first stroke is unitary, $S(\rho_{SA}^{(1)}) = S(\rho_S) + S(\rho_A) = S(\rho_S)$, as the ancillas are taken to be in a pure state. Therefore, the entropy production (12) can be written as

$$\Sigma_H = \Delta S_S + S\left(\rho_A^{(3)}\right) - \mathcal{I}^{(3)}(S:A) - \beta_H Q_H \geqslant 0. \tag{13}$$

This is an important result. The first term is the total change in entropy of the system, $\Delta S_S = S(\rho_S') - S(\rho_S)$. It is precisely one of the goals of the QECC to minimize ΔS_S. The second term in Eq. (13) is the entropy increase in the ancillas. As a byproduct of the QECC, the ancillas become dirty, which is precisely quantified by this term. Hence, $S(\rho_A^{(3)})$ will be exactly the amount of entropy that has to be cleaned up in the last recycling stroke.

The third term in Eq. (13) is the *residual* mutual information that still remains between system and ancilla after the decoding and correction stroke. In the limit of perfect correction, the system would return to ρ_S, so that $\mathcal{I}^{(3)}(S{:}A) = 0$. Hence, $\mathcal{I}^{(3)}(S{:}A)$ represents the shared information that remained in the state $\rho_{SA}^{(3)}$ which the correcting scheme was unable to remove. This mutual information appears with a negative sign, hence contributing to make the process more reversible. The reason for this lies in the fact that before the recycling stroke, $\mathcal{I}^{(3)}(S:A)$ is still, in principle, accessible. As we shall see below, once one includes the recycling stroke, however, these correlations are irretrievably lost.

Finally, the last term in Eq. (13) is the heat flow to the hot bath. Since $Q_H = Q_H^S + Q_H^A$, we may also write (13) more symmetrically as

$$\Sigma_H = \left(\Delta S_S - \beta_H Q_H^S\right) + \left(S\left(\rho_A^{(3)}\right) - \beta_H Q_H^A\right) - \mathcal{I}^{(3)}(S{:}A), \tag{14}$$

which is clearly split into two local contributions, plus a genuinely nonlocal term.

E. Second law for the recycling stroke

One can similarly write the second law for the interaction with the cold bath. In this case, however, an equation of the form (12) would give diverging results, as $\beta_C = \infty$. This pathological behavior of the entropy production in the limit of zero temperature is a known issue, which was discussed, for instance, in Refs. [19–21]. To circumvent, one must provide additional details on the environment interaction generating the map. We therefore assume that each ancilla A_i is coupled to a corresponding environment E_i (not necessarily qubits) prepared in a pure state $|0\rangle_{E_i}$, while the system S is not coupled to anything. We assume in this stroke that the ancillas are fully reset back to $|0\rangle_{A_i}$, which means that each A_iE_i interaction must have the form of a full SWAP. With this proviso, the recycling stroke may be written as the map composition,

$$\rho_{SA}^{(4)} = \mathcal{E}_C^{(A_1)} \otimes \cdots \otimes \mathcal{E}_C^{(A_N)}\left(\rho_{SA}^{(3)}\right), \tag{15}$$

where

$$\mathcal{E}_C^{(A_i)}(\rho) = \mathrm{tr}_{E_i}\left\{U_{A_i,E_i}^{\mathrm{SWAP}}\left(\rho \otimes |0\rangle\langle 0|_{E_i}\right)\left(U_{A_i,E_i}^{\mathrm{SWAP}}\right)^\dagger\right\} \tag{16}$$

is the Stinespring dilation for a map acting only on ancilla A_i.

With this specific representation for the recycling stroke, we can now propose a formula for the entropy production. Namely, based on Refs. [13,16,17], we define the entropy production as being only the mutual information between SA and the cold environment E,

$$\Sigma_C = \mathcal{I}(SA:E) = S\big(\rho_{SA}^{(4)}\big) + S(\rho_E') - S(\rho_{SAE}'), \tag{17}$$

where ρ_{SAE}' denotes the global state of system, ancillas, and cold environment after the map, with ρ_E' being the corresponding reduced density matrix. Within the context of dilated unitary maps, entropy production is often defined with an additional term, proportional to the relative entropy between the initial and final states of the environment [13]. In fact, quite recently, this extra term was shown to be extremely important in a large variety of models [22]. However, in the case of zero temperature, it gives a diverging result since the initial state of the environment is pure. The expression (17), which is also discussed in [16,17], is a choice that does not suffer from this pathology.

Since the global SAE dynamics is unitary, it follows that $S(\rho_{SAE}') = S(\rho_{SA}^{(3)}) + S(\rho_E) = S(\rho_{SA}^{(3)})$. Moreover, we are assuming full thermalization so that $S(\rho_{SA}^{(4)}) = S(\rho_S^{(3)})$. And, finally, again because of the assumption of full thermalization, $S(\rho_E') = S(\rho_A^{(3)})$. Therefore, we conclude that Eq. (17) may also be written as

$$\Sigma_C = S\big(\rho_S^{(3)}\big) + S\big(\rho_A^{(3)}\big) - S\big(\rho_{SA}^{(3)}\big) = \mathcal{I}^{(3)}(S{:}A). \tag{18}$$

This result shows that the entropy production in the cold stroke is simply the residual mutual information that was developed between the system and ancillas in the previous strokes, and which is lost due to the action of the cold bath. This is the same residual mutual information that appears in Eq. (14).

Combining Eqs. (14) and (18) then finally leads to a formula for the total entropy production in the QECC engine,

$$\begin{aligned}\Sigma &= \Sigma_H + \Sigma_C \\ &= \Delta S_S - \beta_H Q_H^S + S\big(\rho_A^{(3)}\big) - \beta_H Q_H^A.\end{aligned} \tag{19}$$

Therefore, the total entropy production is found to contain only *local* contributions, referring to the changes taking place in the system and ancilla.

III. CLASSICAL ERROR-CORRECTING ENGINE

In the remainder of the paper, we apply our results to two specific QECCs. To start, we consider the particularly illuminating case of *classical* error correction. That is, we first consider the protection of diagonal states (in the computational basis) of the form

$$\rho_S = (1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|, \quad p \in [0,1]. \tag{20}$$

This state can be regarded as classical as far as the amplitude damping channel is concerned since, in the sense of einselection [23,24], the amplitude damping chooses the computational basis as a preferred basis.

The effects of the amplitude damping on the state (20) can be corrected by the three-qubit majority voting scheme shown in Fig. 2(a). The encoding unitary U_e is composed of a

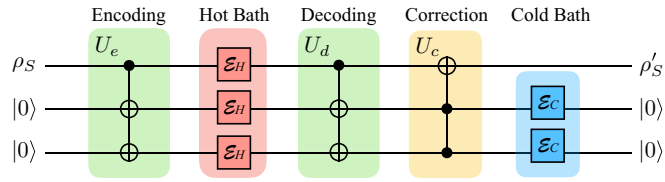

FIG. 2. The classical error-correcting algorithm capable of correcting for diagonal states of the system.

double CNOT,

$$U_e = |0\rangle\langle 0|_S \otimes I_{A_1} \otimes I_{A_2} + |1\rangle\langle 1|_S \otimes X_{A_1} \otimes X_{A_2}, \tag{21}$$

where $X = \sigma_x$ is the Pauli operator. Moreover, the decoding and correction unitary U_{dc} in this case is factored into a product of two terms, $U_{dc} = U_d U_c$, with $U_d = U_e$ and U_c being a Toffoli gate.

All strokes can be computed using standard symbolic algebra. We begin by considering the fidelity between the final and initial states of the system, with and without the QECC. In this case, we assume for simplicity that $\omega = \Omega$, so we can set $f_\Omega = f_\omega \equiv f$. If no QECC is applied, we find, to leading order in the noise strength γ,

$$F(\mathcal{E}_H(\rho_S), \rho_S) \simeq 1 - \frac{\gamma^2}{4(1-p)p}(f-p)^2. \tag{22}$$

As expected, the fidelity is unity if $p = f$, in which case the system already starts with the same population as the environment. Conversely, if the QECC is applied to protect the system, one finds that

$$F(\rho_S', \rho_S) \simeq 1 - \frac{9\gamma^4}{4(1-p)p}[p(1-2f) - f^2(1-2p)]^2. \tag{23}$$

We see that the leading term in the fidelity when the QECC is applied now becomes $\sim\gamma^4$, as compared to γ^2 without the QECC. This neatly shows how error correction behaves at the ensemble level.

The formula for $F(\rho_S', \rho_S)$ when $\omega \neq \Omega$ is more cumbersome. However, here we discuss two limiting cases. First, if the ancillas are gapless, $\omega \to 0$, we find

$$F(\rho_S', \rho_S) = 1 - \frac{(3-2f_\Omega)^2\gamma^4 p}{4(1-p)}. \tag{24}$$

On the other hand, if the ancilla's gap ω is much larger than T, we get

$$F(\rho_S', \rho_S) = 1 - \frac{\gamma^4(1+4f_\Omega-6p)^2}{64p(1-p)}. \tag{25}$$

It is not easy, however, to compare these two limiting cases with the resonant case (23). In fact, a numerical analysis shows that depending on the values of p, f_Ω, and γ, each of the three fidelities can be larger than the other two. Or, putting it differently, the question of which ancillary gap ω yields optimal fidelity does not have a universal answer and depends on the details of all involved parameters. For this reason, we now return to the case $\omega = \Omega$ for simplicity.

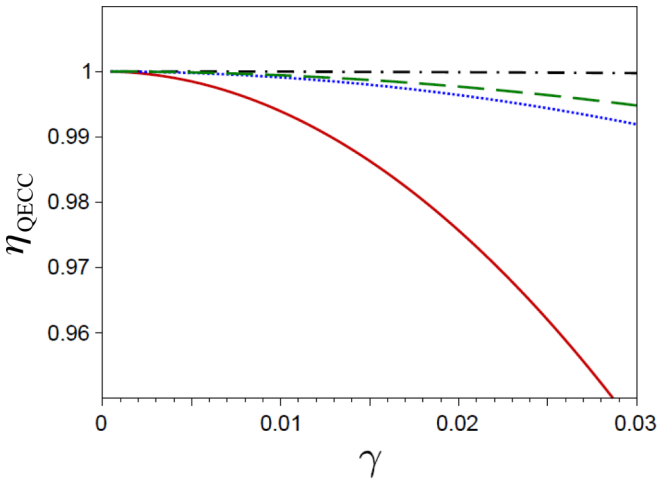

FIG. 3. Efficiency for states with $p = 0.01$ (dash-dotted line), $p = 0.99$ (dashed line), $p = 0.5$ (dotted line), and $p = 0.25$ (solid line). For all cases, $f = 0.2$.

Let us now compute the efficiency defined in Eq. (8). As a proper distance measure, we use the Bures distance squared,

$$D^2(\rho,\sigma) = 2(1-\sqrt{F(\rho,\sigma)}). \tag{26}$$

The efficiency (8), to leading order in γ, then becomes

$$\eta_{\rm QECC} \simeq 1 - \frac{9\gamma^2}{(f-p)^2}[p(1-2f) - f^2(1-2p)]^2. \tag{27}$$

We see that error correction becomes problematic when $p \to f$, as in this case the effect of the channel becomes trivial, so that there is no error to correct. Figure 3 shows the behavior of the efficiency using Eq. (27).

Next we present the heat and work in each step, which we divide into contributions from the system and from the ancillas. The contributions referring to the system, again, to leading order in γ, are

$$W_e^S = 0, \tag{28}$$

$$Q_H^S = \gamma\Omega(f_\Omega - p), \tag{29}$$

$$W_d^S = 0, \tag{30}$$

$$W_c^S \simeq -\gamma\Omega(f_\Omega - p), \tag{31}$$

whereas the contributions from the ancillas are

$$W_e^A = 2p\omega, \tag{32}$$

$$Q_H^A = 2\gamma\omega(f_\omega - p), \tag{33}$$

$$W_d^A \simeq -2p\omega + 2\gamma\omega[f_\Omega + p(3-2f_\omega - 2f_\Omega)], \tag{34}$$

$$W_c^A = 0, \tag{35}$$

$$Q_C^A \simeq -2\gamma\omega[f_\Omega + p(3-2f_\omega-2f_\Omega)] - 2\gamma\omega(f_\omega - p). \tag{36}$$

The physics behind each term is quite interesting. First, the work W_e of the encoding stroke is only associated with the cost of putting the two ancillas in the excited state with probability p. Next, the heat that flows to the hot bath is proportional to the population mismatch $f_\Omega - p$ and $f_\omega - p$. It may thus have any sign depending on the initial value of p. Hence, it is very well possible for heat to flow from SA to the hot bath, and not otherwise.

Particularly interesting is now the analysis of the decoding and correction strokes, W_d and W_c. The decoding work W_d has a zeroth-order contribution from the ancillas, which is *minus* the encoding work, $2p\omega$. If there was no noise, then the process would be entirely reversible. However, due to the hot bath, a new contribution appears. This new contribution, however, appears only in the ancilla, as $W_d^S = 0$. Moreover, this new term is always non-negative since positive temperatures imply $f \in [0, 1/2]$. Hence, we see the total work of the encoding and decoding process, $W_e + W_d > 0$. It *costs* work to encode and decode information when this information is scrambled by the GAD.

The correction work W_c, on the other hand, is seen to be related only to changes in the system and is precisely minus the heat flow Q_H^S between the system and hot bath. As a consequence, the total work performed in one cycle, $W_{\rm tot} = W_e + W_d + W_c$, will be

$$W_{\rm tot} = \Omega\gamma(p - f_\Omega) + 2\gamma\omega[p(3-2f_\omega) + f_\Omega(1-2p)]. \tag{37}$$

The second term is always non-negative, but the first term may have any sign whatsoever. And the step responsible for this is the correction stroke. Thus, while it always costs work to encode and decode information, correcting the state may lead to either a surplus or a deficit of work.

To linear order in γ, it follows that $W_e + Q_H + W_d + W_c + Q_C \simeq 0$. Referring to the first law in Eq. (9), this does not mean that the process is cyclic. Instead, it means that the first nonzero contribution to ΔU_S is of the order of γ^2:

$$\Delta U_S = \gamma^2\Omega\{f_\omega(f_\omega + 4p - 2pf_\omega) + 2f_\Omega(f_\omega + p - 2pf_\omega) - 3p\}. \tag{38}$$

Thus, even though heat and work are all of the order of γ, their net effect only contributes to the total change in energy with a term of the order of γ^2.

IV. SHOR'S NINE-QUBIT CODE

The three-qubit error-correcting scheme considered in the previous section is only capable of correcting diagonal states in the computational basis. Coherences in this basis are not correctly processed. A code which is capable of correcting both incoherent and coherent contributions is Shor's famous nine-qubit code shown in Fig. 4 [25] (see [26] for the implementation without syndrome measurements). This code is quite similar in spirit to the classical code in Fig. 2. The key difference, however, is that the coherent components of ρ_S are also properly encoded due to the application of the Hadamard gates $H = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right)$. Moreover, notice that now the decoding and correction strokes get mixed together, which we separate in Fig. 4 with different colors.

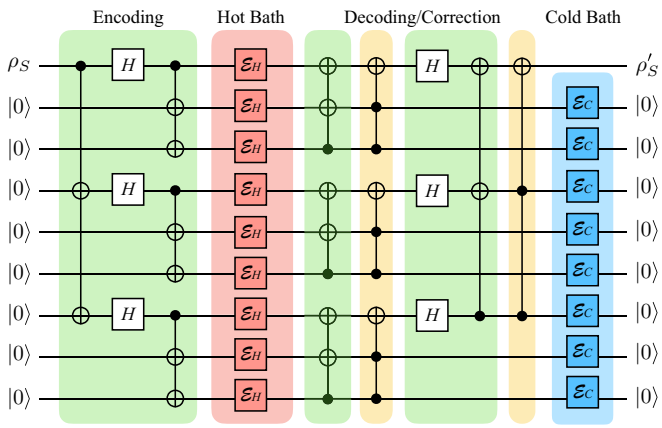

FIG. 4. Shor's nine-qubit code, capable of correcting both the diagonal and the coherent parts of ρ_S against any kind of noise.

We consider a general quantum state of the system, parametrized in the form

$$\rho_S = \begin{pmatrix} p & z\sqrt{p(1-p)} \\ z^*\sqrt{p(1-p)} & 1-p \end{pmatrix}, \quad |z| \leqslant 1. \tag{39}$$

The state is pure when $|z| = 1$. For simplicity, we shall also assume that $\omega = \Omega$, as the calculations become much more complex in this case.

The work and heat in each stroke, for the system and ancilla, are

$$W_e^S = \frac{\Omega}{2}(1-2p), \tag{40}$$

$$Q_H^S = -\frac{\gamma\Omega}{2}(1-2f), \tag{41}$$

$$W_d^S \simeq -\frac{\Omega}{2}(1-2p) + \frac{3\gamma\Omega}{4}(1-2p), \tag{42}$$

$$W_c^S \simeq -\frac{\gamma\Omega}{4}(1+4f-6p), \tag{43}$$

and

$$rClW_e^A = 4\Omega, \tag{44}$$

$$Q_H^A = -4\gamma\Omega(1-2f) \tag{45}$$

$$W_d^A \simeq -4\Omega + 6\gamma\Omega(2-f) \tag{46}$$

$$W_c^A \simeq \gamma\Omega(1-2f) \tag{47}$$

$$Q_C^A \simeq -9\gamma\Omega. \tag{48}$$

Several comments are worth making about these results, particularly when comparing them with the classical results in Eqs. (28)–(36).

First and foremost, we see that all results are independent of the coherences z in Eq. (39). The reason for this is twofold. First, the GAD is a thermal operation and therefore processes populations and coherences independently [20,27]. Second, the Hadamard gates in the encoding and decoding strokes (cf. Fig. 4) acts in a way such that z is not present in the reduced density matrices of a single qubit. Hence, since all thermodynamic quantities involve local Hamiltonians, z does not appear at all in the thermodynamic aspects of the code.

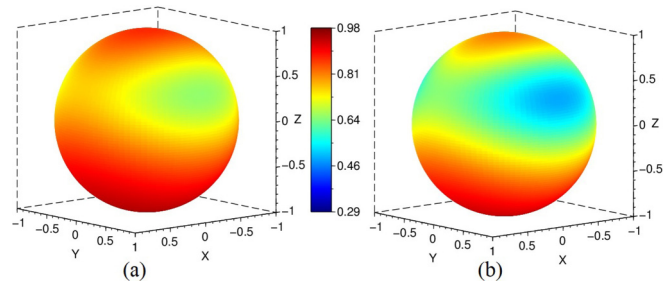

FIG. 5. Shor's code efficiency for different choices of initial pure states for the system, with $f = 0.2$, (a) $\gamma = 0.02$, and (b) $\gamma = 0.03$. The axes refer to different positions of the initial state in the Bloch sphere.

Starting with the encoding stroke, we now see that it requires work in both the system and ancillas to encode information. Moreover, the work cost in the ancillas is entirely independent of the state of the system: for input state ρ_S, it will always cost the same amount, $W_e^A = 4\Omega$, to encode the data in the ancillas (the work cost in the system still depends on p). A similar, but perhaps even more surprising result is that the heat to the hot bath, for *both* the system and ancilla, is entirely independent of the state of the system (this state is exactly true and not to leading order in γ). The heat flow is simply $-\gamma\Omega(1-2f)/2$ per qubit. This is again a consequence of the dramatic influence of the Hadamard gates in Shor's code, which makes it so that after the encoding stroke, the reduced density matrices of all qubits are simply the identity.

The work cost of decoding is similar to the classical case [compare Eqs. (42) and (46) with Eqs. (30) and (34)]: there is a zeroth-order contribution in γ, which is simply the reverse of the encoding work (again representing the reversible part of the process). We also see once again that the correction work can have any sign, as in the classical case. And, finally, we find that the heat to the cold bath is again entirely independent of the state of the system.

On the other hand, the efficiency for the Shor correcting code defined in Eq. (8) depends on the state, as shown in Figs. 5, for pure states ($|z| = 1$), represented as points of the Bloch sphere.

V. DISCUSSIONS AND CONCLUSIONS

The framework of operator error correction (Fig. 1) is formally equivalent to the cyclic operation of a heat engine. This is interesting since it offers an additional link between thermodynamics and information. In this paper, we aimed to explore this connection, by putting forth a thermodynamic analysis of four-stroke codes, which parallel an Otto engine. We emphasize, once again, that QECC and heat engines have entirely different goals. In particular, for operation of a QECC, the work cost is only a marginal concern, as this is marginal compared to the energetics of any real experimental setup. That being said, the directions in which energy flows *are* indeed important. Our analysis shows, for instance, that heat may very well flow *from the system to the hot bath*, something which is counterintuitive. Indeed, this is a common miscon-

ception: neither entropy nor heat have a well-defined sign. What does have a well-defined sign is the *entropy production*, given by Eq. (12).

Another interesting aspect of this thermodynamic analysis is the interplay between the encoding and decoding work costs. The decoding is always the reverse of the encoding operation. But the effect of the noise channel in the middle of the two steps makes the process irreversible, as it scrambles information. As a consequence, there is always a work cost associated with the encoding and decoding steps. This therefore provides a thermodynamic interpretation to an informational task.

Finally, we mention an alternative perspective of the problem. In our formulation, the working fluid was taken to be composed of both the system and ancillas, which then interacted with a hot and a cold bath. Alternatively, one may interpret the system only as the working fluid and the ancillas as a finite-size cold bath. The problem with this formulation is that the system would then interact twice with this cold bath, which leads to questions related to non-Markovianity. The formulation as presented here is more fitting of an actual engine.

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