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PHASE-SPACE MEASURES OF IRREVERSIBILITY

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SUMMARY

- In thermodynamics, irreversibility is quantified by the **entropy production**.
- Goal: to develop quantifiers of irreversibility for quantum systems in contact with a reservoir.
- Idea: move to phase space (bosonic and spin coherent states)
- Motivation:
    - Works at $T = 0$ (current approaches do not).
    - Extensible to multiple reservoirs.
    - Extensible to non-equilibrium baths (e.g. squeezing, dephasing).
    - Gives a microscopic interpretation of entropy production in terms of probability currents.

QUANTIFYING IRREVERSIBILITY

- The energy satisfies a continuity equation:

$$\frac{d\langle H\rangle}{dt} = -\Phi_E$$

- But the entropy does not:

$$\frac{dS}{dt} = \Pi - \Phi$$

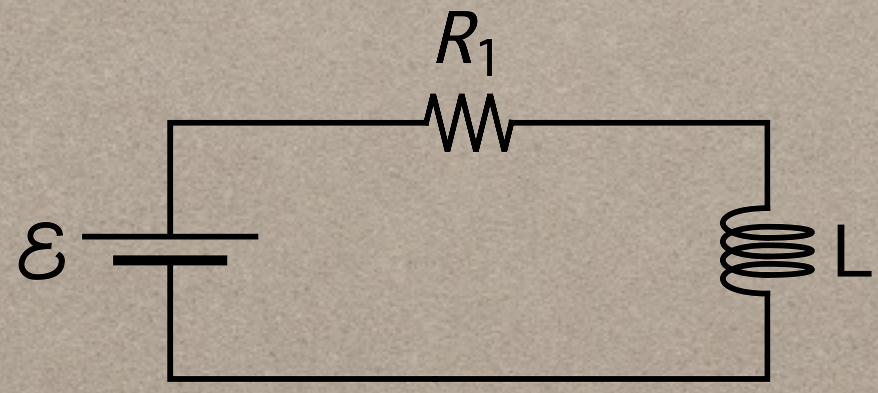
- Π is the entropy production rate

$$\Pi \geq 0 \qquad \text{and} \qquad \Pi = 0 \quad \text{iff we are in equilibrium}$$

STANDARD APPROACH

$$\frac{dS}{dt} = \Pi - \Phi$$

- Von Neumann entropy: $S = -\mathrm{tr}(\rho \ln \rho)$
- Entropy production rate: $\Pi = -\frac{d}{dt} S(\rho || \rho_{\mathrm{eq}})$
- Entropy flux rate: $\Phi = \frac{\Phi_E}{T} \quad \left(\text{“}dS = \frac{dU}{T}\text{”} \right)$



Steady-state

$$\frac{dS}{dt} = 0 \qquad \Pi_{\mathrm{ss}} = \Phi_{\mathrm{ss}} = \frac{\mathcal{E}^2}{RT}$$

Schnakenberg 1976, Spohn 1978, Breuer 2003, Deffner and Lutz, 2011

EXAMPLE: QHO IN A COHERENT STATE

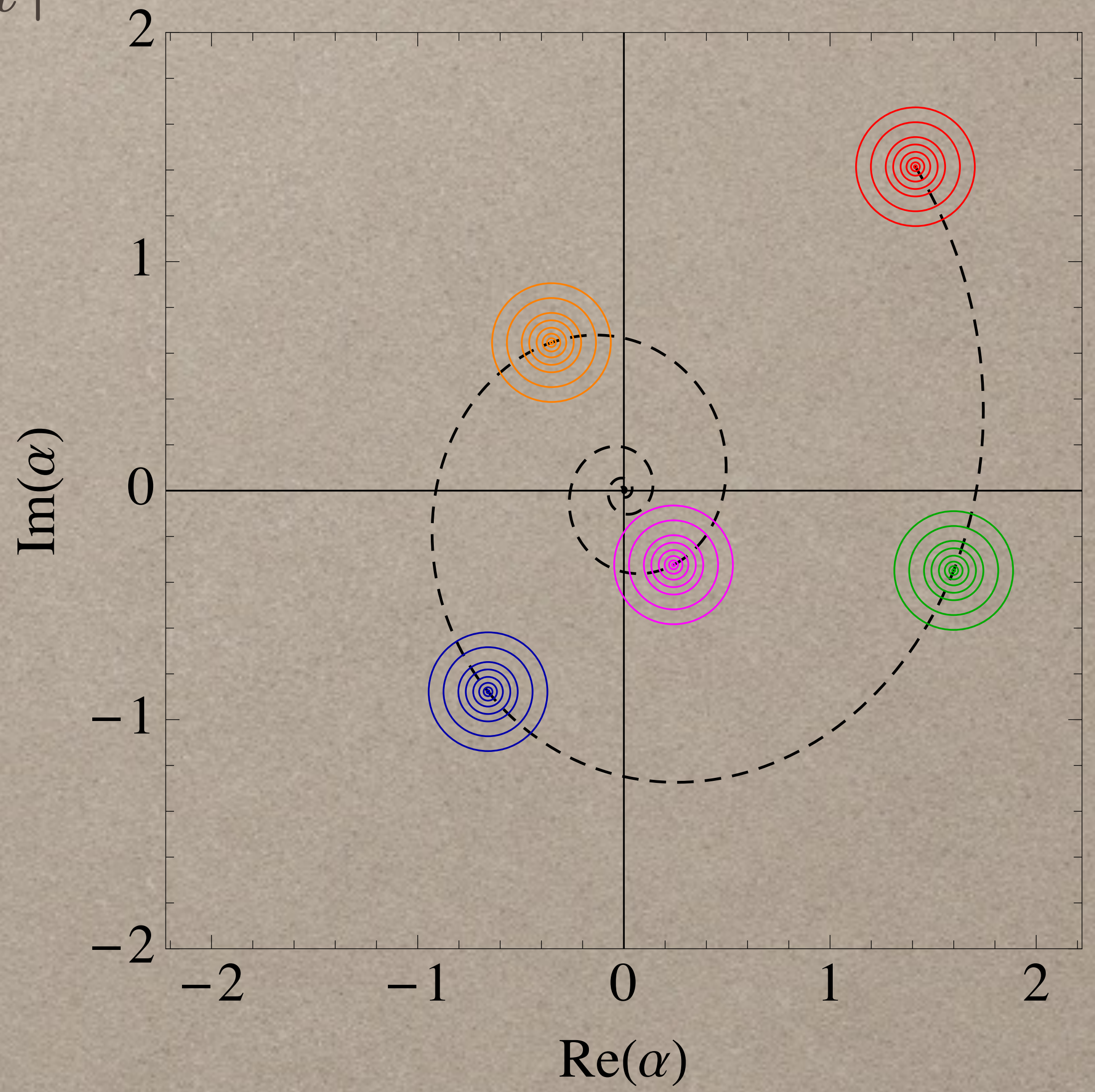
$$\frac{d\rho}{dt} = -i\omega[a^\dagger a, \rho] + \gamma(2a\rho a^\dagger - \{a^\dagger a, \rho\})$$

$$\rho(0) = |\mu\rangle\langle\mu| \;\rightarrow\; \rho(t) = |\mu_t\rangle\langle\mu_t|$$

$$\mu_t = \mu\, e^{-(i\omega+\gamma)t}$$

- Gaussian shape is preserved throughout:

$$\frac{dS}{dt} = 0 \quad \text{but} \quad \Pi = \Phi = \infty$$



PROPOSAL: WIGNER ENTROPY

- Instead of using the von Neumann entropy, we will use the Wigner function and consider entropic measures in phase space.

- Consider a single bosonic mode and define:

$$W(\alpha,\alpha^*) = \frac{1}{\pi^2}\int d^2\lambda e^{-\lambda\alpha^*+\lambda^*\alpha}\mathrm{tr}\left\{\rho e^{\lambda a^\dagger-\lambda^* a}\right\}$$

- We then define the Wigner entropy:

$$S = -\int d^2\alpha W(\alpha,\alpha^*)\ln W(\alpha,\alpha^*)$$

- Note: It was shown in PRL 109, 190502 (2012) that for Gaussian states this coincides with the Rényi-2 entropy

$$S_2 = -\ln \mathrm{tr}\rho^2$$

QUANTUM FOKKER-PLANCK EQUATION

$$\frac{d\rho}{dt} = -i\omega[a^\dagger a, \rho] + D(\rho)$$

$$\bar{n} = \frac{1}{e^{\beta\omega} - 1}$$

$$D(\rho) = \gamma(\bar{n}+1)\left[a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}\right] + \gamma\bar{n}\left[a^\dagger \rho a - \frac{1}{2}\{aa^\dagger, \rho\}\right]$$

- In phase space we get:

$$\partial_t W = -i\omega\left[\partial_{\alpha^*}(\alpha^* W) - \partial_\alpha(\alpha W)\right] + \mathcal{D}(W)$$

$$\mathcal{D}(W) = \partial_\alpha J(W) + \partial_{\alpha^*} J^*(W)$$

$$J(W) = \frac{\gamma}{2}\left[\alpha W + (\bar{n} + 1/2)\partial_{\alpha^*} W\right]$$

J(W) is an irreversible current

$$J(W_{\mathrm{eq}}) = 0$$

HOW TO DEFINE THE WIGNER ENTROPY PRODUCTION RATE

- Next we want to separate: $\frac{dS}{dt} = \Pi - \Phi$
- We did this in three equivalent ways

1. Analogously to the standard formulation:
$$\Pi = -\frac{d}{dt} S(W || W_{\text{eq}})$$
2. Via a manual separation: Φ should be linear in the currents and Π should be quadratic.
3. As an average over stochastic trajectories.

RESULTS

- As a result of this separation, we get:

$$\Pi = \frac{4}{\gamma(\bar{n}+1/2)} \int d^2\alpha \frac{|J(W)|^2}{W}$$

- For Gaussian states, the entropy production is always non-negative and zero only when J(W) = 0 (i.e., iff we are in equilibrium).

- The entropy flux, on the other hand, becomes

$$\Phi = \frac{\gamma}{\bar{n}+1/2}\left[\langle a^\dagger a\rangle - \bar{n}\right]$$

ENTROPY FLUX AND ENERGY FLUX

- Compare with the energy flux

$$\Phi = \frac{\gamma}{\bar{n} + 1/2}\left[\langle a^\dagger a \rangle - \bar{n}\right] \qquad \Phi_E = \gamma\omega\left[\langle a^\dagger a \rangle - \bar{n}\right]$$

- Thus the entropy flux and energy flux will be related by

$$\Phi = \frac{\Phi_E}{\omega(\bar{n} + 1/2)}$$

- At high temperatures $\omega(\bar{n} + 1/2) \simeq T$ so we get

$$\Phi = \frac{\Phi_E}{\omega(\bar{n} + 1/2)} \simeq \frac{\Phi_E}{T}$$

- But now both Π and Φ remain finite at T = 0.

STOCHASTIC TRAJECTORIES AND FLUCTUATION THEOREMS

- We can also arrive at the same result using a completely different method.
  - We analyze the stochastic trajectories in the complex plane.
- The quantum Fokker-Planck equation is equivalent to a Langevin equation in the complex plane:

$$\frac{dA}{dt} = -i\omega A - \frac{\gamma}{2}A + \sqrt{\gamma(\bar{n}+1/2)}\xi(t)$$

$$\langle \xi(t)\xi(t')\rangle = 0, \qquad \langle \xi(t)\xi^*(t')\rangle = \delta(t-t')$$

- We can now define the entropy produced in a trajectory as a functional of the path probabilities for the forward and reversed trajectories:

$$\Sigma[\alpha(t)] = \ln \frac{\mathcal{P}[\alpha(t)]}{\mathcal{P}_R[\alpha^*(\tau - t)]}$$

- This quantity satisfies a fluctuation theorem

$$\langle e^{-\Sigma} \rangle = 1$$

- We show that we can obtain exactly the same formula for the entropy production rate if we define it as

$$\Pi = \frac{\langle d\Sigma[A(t)] \rangle}{dt}$$

DEPHASING BATH

- We also considered the harmonic oscillator dephasing bath

$$D(\rho) = \lambda\left[a^\dagger a \rho a^\dagger a - \frac{1}{2}\{(a^\dagger a)^2, \rho\}\right]$$

- For this bath, applying a similar procedure we find that

$$\Pi = \frac{2}{\lambda}\int \frac{d^2\alpha}{|\alpha|^2}\frac{|I(W)|^2}{W}, \qquad \Phi = 0$$

$$I(W) = \lambda\alpha(\alpha^*\partial_{\alpha^*}W - \alpha\partial_\alpha W)/2$$

- Dephasing bath has no entropy flux: matches the idea of a unital map, as one for which the entropy only increases.

SQUEEZED BATH

- A squeezed bath can be represented by the dissipator

$$\begin{aligned}\mathcal{D}_z(\rho) = \gamma(N+1)&\left[a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}\right] \\ +\gamma N&\left[a^\dagger \rho a - \frac{1}{2}\{a a^\dagger, \rho\}\right] \\ -\gamma M_t&\left[a^\dagger \rho a^\dagger - \frac{1}{2}\{a^\dagger a^\dagger, \rho\}\right] \\ -\gamma M_t^*&\left[a \rho a - \frac{1}{2}\{a a, \rho\}\right]\end{aligned}$$

$$N + 1/2 = (\bar{n} + 1/2)\cosh 2r$$

$$M_t = -(\bar{n} + 1/2)e^{i(\theta - 2\omega_s t)}\sinh 2r$$

- Example of a non-equilibrium reservoir.
- Can be used to extend our ideas beyond the equilibrium paradigm.

- For the squeezed bath we find that the entropy production rate is given by

$$\Pi = \frac{4}{\gamma(\bar{n}+1/2)} \int \frac{d^2\alpha}{W} \Big| J_z \cosh r + J_z^* e^{i(\theta - 2\omega_s t)} \sinh r \Big|^2$$

$$J_z(W) = \frac{\gamma}{2}\Big[\alpha W + (N+1/2)\partial_{\alpha^*} W + M_t \partial_\alpha W\Big]$$

- The entropy flux rate is given by

$$\Phi = \frac{\gamma}{\bar{n}+1/2}\left[\cosh(2r)\langle a^\dagger a\rangle - \bar{n} + \sinh^2(r) - \frac{\mathrm{Re}[M_t^*\langle aa\rangle]}{\bar{n}+1/2}\right]$$

arXiv:1706.01145 (PRL, 118.220601), 2017

ENTROPY PRODUCTION AND LOSS OF COHERENCE

arXiv:1707.08946, 2017

Baumgratz, Cramer and Plenio, *Quantifying coherence. PRL,* ***113****,*
140401 (2014)

- Consider a system in contact with a bath, initially prepared in a non-equilibrium state.
- The non-equilibrium free energy F = U - T S, may be written as

$$F(\rho) = F_{\mathrm{eq}} + TS(\rho||\rho_{\mathrm{eq}})$$

- Since $S(\rho||\rho_{\mathrm{eq}}) \geq 0$ then F ≥ Feq
- We can now define the entropy production rate as

$$\Pi = -\frac{1}{T}\frac{dF}{dt}$$

- Next, choose as preferred basis the Hamiltonian basis (this is the basis imposed by the bath).

- We may then separate

$$S(\rho||\rho_{\mathrm{eq}}) = S(p||p_{\mathrm{eq}}) + C(\rho)$$

$$S(p||p_{\mathrm{eq}}) = \sum_n p_n \ln p_n / p_n^{\mathrm{eq}}$$

$$C(\rho) = S(\rho_d) - S(\rho)$$

- The non-eq. free energy then becomes:

$$F(\rho) = F_{\mathrm{eq}} + T S(p||p_{\mathrm{eq}}) + T C(\rho)$$

- We therefore see that the entropy production separates into two terms
  - Entropy is produced because the system undergoes transitions in the energy levels in order to equilibrate with the bath.
  - But entropy is also produced because the bath is destroying coherences: $\Pi = \Pi_d + \Upsilon$

$$\Pi_d = -\frac{1}{T}\frac{d}{dt}S(p||p_{\text{eq}}) \qquad\qquad \Upsilon = -\frac{1}{T}\frac{dC}{dt}$$

Note: entropy flux is not affected by coherences! $\Phi = \dfrac{\Phi_E}{T}$

PHASE SPACE VIEW OF DECOHERENCE

- Now we move once again to phase space and try to observe this effect in terms of microscopic currents.

- For this we move to spin systems using spin coherent states for a spin *J*:

$$|\Omega\rangle = e^{-\phi J_z} e^{-\theta J_y} |J, J\rangle$$

$$\mathcal{Q}(\Omega) = \langle \Omega | \rho | \Omega \rangle$$

Spin Husimi function

$$\Sigma = -\int d\Omega \, \mathcal{Q}(\Omega) \ln \mathcal{Q}(\Omega)$$

Wehrl entropy

DEPHASING BATH

- We start with the dephasing bath, which induces no population changes.

$$D(\rho) = -\frac{\lambda}{2}[J_z, [J_z, \rho]]$$

- Again, there is no entropy flux: $\Phi = 0$

- As for the entropy production, we get:

$$\Pi = \frac{\lambda}{2} \int d\Omega \frac{|\mathcal{J}_z(\mathcal{Q})|^2}{\mathcal{Q}} \qquad \mathcal{J}_z(\mathcal{Q}) = -i\partial_\phi \mathcal{Q}$$

AMPLITUDE DAMPING

- For the amplitude damping, we have both processes (decoherence and energy level transitions).

$$D(\rho) = \gamma(\bar{n}+1)\left[J_-\rho J_+ - \frac{1}{2}\{J_+J_-,\rho\}\right] + \gamma\bar{n}\left[J_+\rho J_- - \frac{1}{2}\{J_-J_+,\rho\}\right]$$

- We now get, for the entropy production rate

$$\Pi = \frac{\gamma}{2}\int \frac{d\Omega}{\mathcal{Q}}\Bigg\{ \frac{[2J\mathcal{Q}\sin\theta + (\cos\theta - (2\bar{n}+1))\partial_\theta \mathcal{Q}]^2}{(2\bar{n}+1)-\cos\theta}$$

$$+|\mathcal{J}_z(\mathcal{Q})|^2\left[(2\bar{n}+1)\cos\theta - 1\right]\frac{\cos\theta}{\sin^2\theta}\Bigg\}$$

CONCLUSIONS

- Entropy production quantifies irreversibility. But you need to separate it from the entropy flux

$$\frac{dS}{dt} = \Pi - \Phi$$

- Phase space formulation allows one to work at T = 0 and work with non-equilibrium baths.

- Π captures population changes and loss of coherence.

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Thank you!