PHASE-SPACE MEASURES OF IRREVERSIBILITY

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SUMMARY

• In thermodynamics, irreversibility is quantified by the entropy production.

• Goal: to develop quantifiers of irreversibility for quantum systems in contact with a reservoir.

• Idea: move to phase space (bosonic and spin coherent states)

• Motivation:
  • Works at $T = 0$ (current approaches do not).
  • Extensible to multiple reservoirs.
  • Extensible to non-equilibrium baths (e.g. squeezing, dephasing).
  • Gives a microscopic interpretation of entropy production in terms of probability currents.
QUANTIFYING IRREVERSIBILITY

• The energy satisfies a continuity equation:

\[
\frac{d \langle H \rangle}{dt} = -\Phi_E
\]

• But the entropy does not:

\[
\frac{dS}{dt} = \Pi - \Phi
\]

• \( \Pi \) is the entropy production rate

\[ \Pi \geq 0 \quad \text{and} \quad \Pi = 0 \quad \text{iff we are in equilibrium} \]
STANDARD APPROACH

- Von Neumann entropy:
  \[ S = -\text{tr} (\rho \ln \rho) \]

- Entropy production rate:
  \[ \Pi = -\frac{dS}{dt} (\rho \| \rho_{\text{eq}}) \]

- Entropy flux rate:
  \[ \Phi = \frac{\Phi_E}{T} \left( \text{“} dS = \frac{dU}{T} \text{”} \right) \]

Steady-state

\[ \frac{dS}{dt} = 0 \quad \Pi_{ss} = \Phi_{ss} = \frac{\mathcal{E}^2}{RT} \]

EXAMPLE: QHO IN A COHERENT STATE

\[
\frac{d\rho}{dt} = -i\omega [a^\dagger a, \rho] + \gamma (2a\rho a^\dagger - \{a^\dagger a, \rho\})
\]

\[
\rho(0) = |\mu\rangle\langle \mu| \rightarrow \rho(t) = |\mu_t\rangle\langle \mu_t|
\]

\[
\mu_t = \mu e^{-(i\omega+\gamma)t}
\]

- Gaussian shape is preserved throughout:

\[
\frac{dS}{dt} = 0 \quad \text{but} \quad \Pi = \Phi = \infty
\]
PROPOSAL: WIGNER ENTROPY

- Instead of using the von Neumann entropy, we will use the Wigner function and consider entropic measures in phase space.

- Consider a single bosonic mode and define:

\[
W(\alpha, \alpha^*) = \frac{1}{\pi^2} \int d^2 \lambda e^{-\lambda \alpha^* + \lambda^* \alpha} \text{tr} \left\{ \rho e^{\lambda a^\dagger - \lambda^* a} \right\}
\]

- We then define the Wigner entropy:

\[
S = - \int d^2 \alpha W(\alpha, \alpha^*) \ln W(\alpha, \alpha^*)
\]

- Note: It was shown in PRL 109, 190502 (2012) that for Gaussian states this coincides with the Rényi-2 entropy

\[
S_2 = - \ln \text{tr} \rho^2
\]
QUANTUM FOKKER-PLANCK EQUATION

\[
\frac{d\rho}{dt} = -i\omega [a^\dagger a, \rho] + D(\rho)
\]

\[
\bar{n} = \frac{1}{e^{\beta\omega} - 1}
\]

\[
D(\rho) = \gamma(\bar{n} + 1) \left[a\rho a^\dagger - \frac{1}{2} \{a^\dagger a, \rho\}\right] + \gamma\bar{n} \left[a^\dagger \rho a - \frac{1}{2} \{aa^\dagger, \rho\}\right]
\]

- In phase space we get:

\[
\partial_t W = -i\omega \left[\partial_{\alpha^*}(\alpha^* W) - \partial_\alpha (\alpha W)\right] + D(W)
\]

\[
D(W) = \partial_\alpha J(W) + \partial_{\alpha^*} J^*(W)
\]

\[
J(W) = \frac{\gamma}{2} \left[\alpha W + (\bar{n} + 1/2)\partial_{\alpha^*}W\right]
\]

\[
J(W) \text{ is an irreversible current}
\]

\[
J(W_{eq}) = 0
\]
HOW TO DEFINE THE WIGNER ENTROPY PRODUCTION RATE

• Next we want to separate: \[ \frac{dS}{dt} = \Pi - \Phi \]

• We did this in three equivalent ways

  1. Analogously to the standard formulation:
     \[ \Pi = -\frac{d}{dt} S(W||W_{eq}) \]

  2. Via a manual separation: \( \Phi \) should be linear in the currents and \( \Pi \) should be quadratic.

  3. As an average over stochastic trajectories.
RESULTS

• As a result of this separation, we get:

\[ \Pi = \frac{4}{\gamma(\bar{n} + 1/2)} \int d^2\alpha \frac{|J(W)|^2}{W} \]

• For Gaussian states, the entropy production is always non-negative and zero only when \( J(W) = 0 \) (i.e., iff we are in equilibrium).

• The entropy flux, on the other hand, becomes

\[ \Phi = \frac{\gamma}{\bar{n} + 1/2} \left[ \langle a^\dagger a \rangle - \bar{n} \right] \]
ENTROPY FLUX AND ENERGY FLUX

• Compare with the energy flux

\[ \Phi = \frac{\gamma}{\bar{n} + 1/2} \left[ \langle a^\dagger a \rangle - \bar{n} \right] \]
\[ \Phi_E = \gamma \omega \left[ \langle a^\dagger a \rangle - \bar{n} \right] \]

• Thus the entropy flux and energy flux will be related by

\[ \Phi = \frac{\Phi_E}{\omega(\bar{n} + 1/2)} \]

• At high temperatures \( \omega(\bar{n} + 1/2) \approx T \) so we get

\[ \Phi = \frac{\Phi_E}{\omega(\bar{n} + 1/2)} \approx \frac{\Phi_E}{T} \]

• But now both \( \Pi \) and \( \Phi \) remain finite at \( T = 0 \).
STOCHASTIC TRAJECTORIES AND FLUCTUATION THEOREMS

• We can also arrive at the same result using a completely different method.

• We analyze the stochastic trajectories in the complex plane.

• The quantum Fokker-Planck equation is equivalent to a Langevin equation in the complex plane:

\[
\frac{dA}{dt} = -i\omega A - \frac{\gamma}{2} A + \sqrt{\gamma(\bar{n} + 1/2)}\xi(t)
\]

\[
\langle \xi(t)\xi(t') \rangle = 0, \quad \langle \xi(t)\xi^*(t') \rangle = \delta(t - t')
\]
• We can now define the entropy produced in a trajectory as a functional of the path probabilities for the forward and reversed trajectories:

\[
\Sigma[\alpha(t)] = \ln \frac{\mathcal{P}[\alpha(t)]}{\mathcal{P}_R[\alpha^*(\tau - t)]}
\]

• This quantity satisfies a fluctuation theorem

\[
\langle e^{-\Sigma} \rangle = 1
\]

• We show that we can obtain exactly the same formula for the entropy production rate if we define it as

\[
\Pi = \frac{\langle d\Sigma[A(t)] \rangle}{dt}
\]
DEPHASING BATH

• We also considered the harmonic oscillator dephasing bath

\[ D(\rho) = \lambda \left[ a^\dagger a \rho a^\dagger a - \frac{1}{2} \{ (a^\dagger a)^2, \rho \} \right] \]

• For this bath, applying a similar procedure we find that

\[ \Pi = \frac{2}{\lambda} \int \frac{d^2 \alpha}{|\alpha|^2} \frac{|I(W)|^2}{W}, \quad \Phi = 0 \]

\[ I(W) = \lambda \alpha (\alpha^* \partial_{\alpha^*} W - \alpha \partial_{\alpha} W) / 2 \]

• Dephasing bath has no entropy flux: matches the idea of a unital map, as one for which the entropy only increases.
SQUEEZED BATH

• A squeezed bath can be represented by the dissipator

\[ \mathcal{D}_z(\rho) = \gamma (N + 1) \left[ apa^\dagger - \frac{1}{2} \{ a^\dagger a, \rho \} \right] \]

\[ + \gamma N \left[ a^\dagger \rho a - \frac{1}{2} \{ aa^\dagger, \rho \} \right] \]

\[ - \gamma M_t \left[ a^\dagger \rho a^\dagger - \frac{1}{2} \{ a^\dagger a^\dagger, \rho \} \right] \]

\[ - \gamma M_t^* \left[ \rho a - \frac{1}{2} \{ aa, \rho \} \right] \]

\[ N + 1/2 = (\bar{n} + 1/2) \cosh 2r \]

\[ M_t = -(\bar{n} + 1/2) e^{i(\theta - 2\omega_s t)} \sinh 2r \]

• Example of a non-equilibrium reservoir.

• Can be used to extend our ideas beyond the equilibrium paradigm.
For the squeezed bath we find that the entropy production rate is given by

\[
\Pi = \frac{4}{\gamma(\bar{n} + 1/2)} \int \frac{d^2\alpha}{W} |J_z \cosh r + J^{*}_z e^{i(\theta - 2\omega_s t)} \sinh r|^2
\]

\[
J_z(W) = \frac{\gamma}{2} \left[ \alpha W + (N + 1/2) \partial_{\alpha^{*}} W + M_t \partial_{\alpha} W \right]
\]

The entropy flux rate is given by

\[
\Phi = \frac{\gamma}{\bar{n} + 1/2} \left[ \cosh(2r)\langle a^{\dagger}a \rangle - \bar{n} + \sinh^2(r) - \frac{\text{Re}[M_t^{\ast}\langle aa \rangle]}{\bar{n} + 1/2} \right]
\]

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ENTROPY PRODUCTION AND LOSS OF COHERENCE

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Baumgratz, Cramer and Plenio, Quantifying coherence. PRL, 113, 140401 (2014)
Consider a system in contact with a bath, initially prepared in a non-equilibrium state.

The non-equilibrium free energy $F = U - TS$, may be written as

$$F(\rho) = F_{eq} + TS(\rho || \rho_{eq})$$

Since $S(\rho || \rho_{eq}) \geq 0$ then $F \geq F_{eq}$

We can now define the entropy production rate as

$$\Pi = -\frac{1}{T} \frac{dF}{dt}$$
• Next, choose as preferred basis the Hamiltonian basis (this is the basis imposed by the bath).

• We may then separate

\[ S(\rho||\rho_{eq}) = S(p||p_{eq}) + C(\rho) \]

\[ S(p||p_{eq}) = \sum_{n} p_n \ln \frac{p_n}{p_{eq}^n} \]

\[ C(\rho) = S(\rho_d) - S(\rho) \]

• The non-eq. free energy then becomes:

\[ F(\rho) = F_{eq} + TS(p||p_{eq}) + TC(\rho) \]
• We therefore see that the entropy production separates into two terms

• Entropy is produced because the system undergoes transitions in the energy levels in order to equilibrate with the bath.

• But entropy is also produced because the bath is destroying coherences: \( \Pi = \Pi_d + \gamma \)

\[
\Pi_d = -\frac{1}{T} \frac{d}{dt} S(p||p_{eq}) \quad \gamma = -\frac{1}{T} \frac{dC}{dt}
\]

Note: entropy flux is not affected by coherences! \( \Phi = \frac{\Phi_E}{T} \)
PHASE SPACE VIEW OF DECOHERENCE

- Now we move once again to phase space and try to observe this effect in terms of microscopic currents.

- For this we move to spin systems using spin coherent states for a spin $J$:

$$ |\Omega\rangle = e^{-\phi J_z} e^{-\theta J_y} |J, J\rangle $$

$$ Q(\Omega) = \langle \Omega | \rho | \Omega \rangle $$

Spin Husimi function

$$ \Sigma = - \int d\Omega Q(\Omega) \ln Q(\Omega) $$

Wehrl entropy
DEPHASING BATH

- We start with the dephasing bath, which induces no population changes.

\[ D(\rho) = -\frac{\lambda}{2} [J_z, [J_z, \rho]] \]

- Again, there is no entropy flux: \( \Phi = 0 \)

- As for the entropy production, we get:

\[ \Pi = \frac{\lambda}{2} \int d\Omega \frac{|J_z(Q)|^2}{Q} \quad J_z(Q) = -i \partial_\phi Q \]
AMPLITUDE DAMPING

- For the amplitude damping, we have both processes (decoherence and energy level transitions).

\[ D(\rho) = \gamma (\bar{n} + 1) \left[ J_- \rho J_+ - \frac{1}{2} \{ J_+ J_-, \rho \} \right] + \gamma \bar{n} \left[ J_+ \rho J_- - \frac{1}{2} \{ J_- J_+, \rho \} \right] \]

- We now get, for the entropy production rate

\[
\Pi = \frac{\gamma}{2} \int \frac{d\Omega}{Q} \left\{ \frac{[2J Q \sin \theta + (\cos \theta - (2\bar{n} + 1)) \partial_{\theta} Q]^2}{(2\bar{n} + 1) - \cos \theta} \right. \\
+ |J_z(Q)|^2 \left[ (2\bar{n} + 1) \cos \theta - 1 \right] \left. \frac{\cos \theta}{\sin^2 \theta} \right\}
\]
CONCLUSIONS

• Entropy production quantifies irreversibility. But you need to separate it from the entropy flux

\[
\frac{dS}{dt} = \Pi - \Phi
\]

• Phase space formulation allows one to work at \( T = 0 \) and work with non-equilibrium baths.

• \( \Pi \) captures population changes and loss of coherence.

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Thank you!