Quantum Information and Quantum Noise - Problem set 1

Professor: Gabriel T. Landi

Deadline: 09/04

1. **Eigenfun!** Let A be a Hermitian matrix with eigenstuff $A|v_i\rangle = \lambda_i|v_i\rangle$.

   (a) Find the eigenvalues and eigenvectors of A^n, in terms of $|v_i\rangle$ and λ_i. Note that this includes A^{-1}.

   (b) Do the same for $A + b$, where b is a constant.

   (c) Do the same for cA, where c is a constant.

   (d) Let $f(x)$ be an arbitrary function defined by a power series expansion $f(x) = \sum_n c_n x^n$. Find the eigenvalues and eigenvectors of $f(A)$ (again in terms of $|v_i\rangle$ and λ_i).

   (e) Let $B = SAS^{-1}$. Relate the eigenvalues and eigenvectors of B and A. This is called a similarity transformation. Unitaries are a particular case, in which $S^{-1} = S^\dagger$.

   (f) We can also invert the argument on our previous result: if A and B are two operators which share the same eigenvalues, then there must exist a similarity transformation between them. Show that if A and B are both Hermitian, then this transformation must be accomplished by a unitary.

   (g) Show that $Sf(A)S^{-1} = f(SAS^{-1})$: similarity transformations infiltrate functions! This type of infiltration appear all the time. For instance, given a unitary U, $Ue^AU^\dagger = e^{UAU^\dagger}$.

   (h) Now let C be an arbitrary (not necessarily Hermitian) *finite dimensional* operator. Show that $C^\dagger C$ and $CC^\dagger$ are positive semi-definite operators[1] and share the same eigenvalues, although in general they are different. Note that this property holds only for finite dimensional systems. For instance, the eigenvalues of $a^\dagger a$ are $0, 1, 2, \ldots$, but the eigenvalues of $aa^\dagger$ are $1, 2, 3, \ldots$.

2. **Functions of operators.** The goal of this exercise is to show you that, when manipulating functions of operators, all that matters is the *algebra*. That is, we don't need to know what are the actual matrix elements or even if the matrix is finite or infinite. All properties follow only from the abstract algebra between operators.

   (a) Let A be an operator such that $A^2 = 1$. Find $e^{\alpha A}$, where α is a constant. Your results contemplate as a particular case the operators σ_x, σ_y and σ_z.

   (b) Let B be an operator such that $B^2 = 0$. Find $e^{\alpha B}$. This now contemplates σ_+ and σ_-.

   (c) Consider now the angular momentum operators $S_{x,y,z}$ satisfying $[S_i, S_j] = i\epsilon_{i,j,k}S_k$. Compute $e^{\alpha S_x} S_z e^{-\alpha S_x}$. Your results hold for arbitrary spin. If $S_i = \sigma_i/2$ (spin 1/2) then you can check your calculation by using the results in (a).

[1] A positive definite operator is one whose eigenvalues are always positive. Semi-definite means the eigenvalues can be either positive or zero (i.e., non-negative). When we say an operator is positive definite or semi-definite it is already implicit that it is also Hermitian, for the very notion of "positive eigenvalues" would not be defined unless the eigenvalues are real.

(d) Consider now the harmonic oscillator/bosonic operators a and $a^\dagger$, which satisfy $[a, a^\dagger] = 1$.[2] Compute the commutation relations of

$$A_+ = \frac{a^\dagger a^\dagger}{2}, \qquad A_- = -\frac{aa}{2}, \qquad A_z = a^\dagger a + \frac{1}{2}. \tag{1}$$

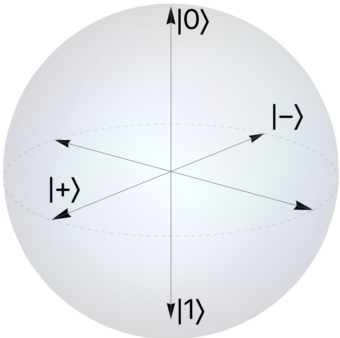
Show that they satisfy the same algebra as σ_+, σ_- and σ_z.

(e) Finally, let $S(r) = \exp\{\frac{r}{2}(a^\dagger a^\dagger - aa)\}$. Use the results from the previous items to show that this operator may be factored as

$$S(r) = e^{\tanh(r) A_+} e^{-\ln(\cosh(r)) A_z} e^{\tanh(r) A_-} \tag{2}$$

This is the message I wanted you to take home: *all that matters is the algebra*. If infinite dimensional operators happen to satisfy the same algebra as 2×2 matrices, lucky for us!

3. **Quantum gates in Bloch's sphere.** Consider a single qubit and the Bloch sphere representation (see figure). The goal of this exercise is to understand what the different points in Bloch's sphere represent and how to move around it by means of unitaries.



(a) A general parametrization of the state of a qubit is given by

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle \tag{3}$$

Show that all states are contained within this parametrization and with $\theta \in [0, \pi)$, $\phi \in [0, 2\pi]$. That is, show that if you choose any state which is not in this set, then you either violate normalization or you get back the same state up to a global phase (which is physically irrelevant).

(b) Consider now the *bit-flip* gate

$$X = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4}$$

Study the action of this gate on the general state $|\psi\rangle$ and, from this, explain intuitively what it does in the Bloch's sphere representation.

[2] You probably know what are annihilation and creation operators of course. But I want to emphasize that for this problem you don't need to know anything about them, except the fact that $[a, a^\dagger] = 1$.

(c) Do the same for the *Haddamard* gate

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{5}$$

(d) Do the same for the *phase-shift* gate

$$V_\chi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\chi} \end{pmatrix} \tag{6}$$

When $\chi = \pi$ we get $V_\pi = Z$, which is also called in this case a *phase-flip* gate.

4. **Amplitude Damping.** In class we saw the effects of the dephasing model in producing decoherence. Here I want you to study the amplitude damping model, which consists of a qubit subject to the following master equation

$$\frac{d\rho}{dt} = -i\frac{\Omega}{2}[\sigma_z, \rho] + \gamma\Big[\sigma_-\rho\sigma_+ - \frac{1}{2}(\sigma_+\sigma_-\rho + \rho\sigma_+\sigma_-)\Big] \tag{7}$$

Study the evolution of $\rho(t)$ starting from a general state (which you can parametrize, like we did in class). The message I want you to have from this exercise, is that for the amplitude damping decoherence also occurs, but it does so together with changes in the populations (diagonal elements).

5. **Bipartite entanglement.** Consider a bipartite system AB prepared in the state

$$|\psi\rangle = \frac{c}{\sqrt{2}}(|0,0\rangle + |1,1\rangle) + \frac{d}{\sqrt{2}}(|0,1\rangle + |1,0\rangle) \tag{8}$$

where c and d are real numbers. If you wish, you can parametrize them in a convenient way using some angle.

(a) Without doing any calculations, try to understand for which values of c and d the state will or will not be entangled.

(b) Compute the reduced density matrices ρ_A and ρ_B.

(c) Compute the purity.

(d) Find the Schmidt decomposition of $|\psi\rangle$.

(e) Compute the von Neumann entropy of ρ_A and ρ_B.

(f) Compute all Rényi-α entropies.

(g) Plot all these things in terms of the angle that you used to parametrize c and d. I want pretty plots![3]

6. **Entangling through interactions.** In order to entangle particles, we must make them interact.

(a) Consider first a single qubit in an arbitrary state ρ. Show that the purity is preserved by any unitary evolution. Associate this with the radius of Bloch's sphere.

[3] This is an example of an open ended problem where I won't tell you exactly what to analyze. Instead, you will have to figure out yourself what are the interesting quantities to look at. I don't do this because I am mean (which I'm not) or lazy (which I definitely am), I do it because that is how things work in research.

(b) Now consider two interacting qubits evolving according to the Hamiltonian

$$H = g(\sigma_+^1 \sigma_-^2 + \sigma_-^1 \sigma_+^2) \tag{9}$$

In the context of continuous variables (which we will cover soon), this is usually called a *beam splitter* interaction. Suppose that they are initially in the product state $|\psi(0)\rangle = |0, 1\rangle_{AB}$. Compute the state at an arbitrary time t.

(c) Compute now the reduced density matrices and the populations of each qubit as a function of time.

(d) Compute the purity (or some entropy) of the reduced state and show that by interacting, the two qubits become entangled. Find the time where the entanglement is maximum.

(e) Sketch the trajectory of qubit A in its Bloch sphere.

7. **Consuming correlations to reverse the flow of heat.**[4] Consider two qubits with individual Hamiltonians $H_A = -\Omega\sigma_z^A/2$ and $H_B = -\Omega\sigma_z^B/2$. Suppose that they are initially prepared in a correlated state

$$\rho_{AB}(0) = \rho_A^0 \otimes \rho_B^0 + \chi \tag{10}$$

where $\rho_i^0 = e^{-\beta_i H_i}/Z_i$, with $Z_i = \mathrm{tr}(e^{-\beta_i H_i})$ and $\beta_i = 1/T_i$ being the temperature in which each qubit was prepared. Moreover, $\chi = \alpha(|0, 1\rangle\langle 1, 0| + |1, 0\rangle\langle 0, 1|)$, where α is a constant. The interesting thing about this state is that $\mathrm{tr}(\chi) = 0$ so, if you look only at A or only at B, they behave like thermal states. But they are nonetheless correlated. Suppose now that the two qubits are allowed to evolve according to the interaction

$$H = g(\sigma_x^A \sigma_y^B - \sigma_y^A \sigma_x^B) \tag{11}$$

Study the flow of energy in this system, characterized, for instance, by $\frac{\mathrm{d}\langle H_A\rangle}{\mathrm{d}t}$. Compare this with the correlation between the two systems, measured by the mutual information $\mathcal{I}_{AB} = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$. I want you to show that if there is no correlation ($\alpha = 0$), heat will flow from hot to cold. But if they are correlated, then the correlation can be consumed to make the heat flow from cold to hot.

[4] This was based on arXiv:1711.03323