Quantum Information and Quantum Noise - Problem set 2

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Deadline: 25/04 (wednesday)

1. **Measurements with two qubits.** Suppose you have your qubit S which is prepared in a certain state $|\psi\rangle$ which you want to probe. In this problem we will analyze one way to do so, by coupling our qubit to another qubit ancilla. We assume the ancilla is initialized in an arbitrary state
$$|\chi\rangle_A = \cos(\chi)|0\rangle_A + \sin(\chi)|1\rangle_A.$$
Then we evolve the composite state $|\psi\rangle_S \otimes |\chi\rangle_A$ with a CNOT unitary[1]
$$U = |0\rangle_S\langle 0| \otimes \mathbb{I}_A + |1\rangle_S\langle 1| \otimes \sigma_x^A$$
Finally, after the evolution we perform a projective measurement on the ancilla in a certain basis:
$$|\phi_0\rangle_A = \cos(\phi)|0\rangle_A + \sin(\phi)|1\rangle_A$$
$$|\phi_1\rangle_A = -\sin(\phi)|0\rangle_A + \cos(\phi)|1\rangle_A$$
   (a) Find the set of generalized measurement operators $\{M_i\}$ associated with this protocol.
   (b) Find the associated POVM $\{E_i\}$.
   (c) Discuss for which values of ϕ and χ we get a projective measurement.
   (d) Discuss for which values of ϕ and χ we do no measurement at all. That is, such that $E_0 = E_1 = \mathbb{I}/2$.

2. **Single-mode squeezing.** The squeezing operator is defined as[2]
$$S_z = \exp\left\{\frac{1}{2}(z a^\dagger a^\dagger - z^* a a)\right\},$$
where $z = re^{i\theta}$ is a complex number. Note that, by construction, S_z is unitary.
   (a) Show that
$$S_z^\dagger a S_z = a\cosh r + a^\dagger e^{i\theta}\sinh r.$$
   (b) Let $|z\rangle = S_z|0\rangle$ denote the *squeezed vacuum*. Find the coefficients expressing this state in the Fock basis. Tip: use the following result,
$$S(z) = e^{\frac{a^\dagger a^\dagger}{2} e^{i\theta}\tanh(r)} e^{-\ln(\cosh(r))(a^\dagger a + 1/2)} e^{-\frac{aa}{2} e^{-i\theta}\tanh(r)} \qquad (1)$$
   which is a slight generalization of problem 2(e) of Problem Set 1.

[1] Just in case you are wondering, this could be generated by a Hamiltonian $H = g(|1\rangle_S\langle 1| \otimes \sigma_y)$ provided we take $gt = \pi/2$.

[2] Be careful, some people define it with $z \to -z$.

(c) Consider now a Hamiltonian of the form

$$H = \omega a^\dagger a + \frac{1}{2}(\lambda a^\dagger a^\dagger + \lambda^* aa).$$

Diagonalize this Hamiltonian by applying the squeezing operator. That is, show that by a suitable choice of z, we can write

$$S_z^\dagger H S_z = \Omega a^\dagger a + \text{const}$$

where Ω is a new constant that depends on ω and λ. Discuss for which values of ω and λ this diagonalization procedure works. What do you think happens if this condition is violated?

(d) Consider now a *displaced squeezed state* $|\alpha, z\rangle = D(\alpha)S_z|0\rangle$ where, recall $z = re^{i\theta}$. Verify that in the limit of infinite squeezing $r \to \infty$ this state becomes the eigenstate of the position and momentum operators q and p, depending on the direction θ. That is, multiply $|\alpha, z\rangle$ by $q = \frac{1}{\sqrt{2}}(a + a^\dagger)$. Then show that in the limit $r \to \infty$, with a smart choice of θ, we get back a number times the same state. What is the relation between the coherent state α and the eigenvalue of q? Do the same for p.

(e) Finally, compute the Husimi function of the squeezed vacuum,

$$\rho = S_z|0\rangle\langle 0|S_z^\dagger$$

Tip: use the decomposition in Eq. (1) to compute $\langle\alpha|S_z|0\rangle$. You will find that your Husimi function is again a Gaussian distribution. One is therefore led to ask what is the most general Gaussian state. It turns out that it is a displaced squeezed thermal state

$$\rho = D(\mu)S_z \frac{e^{-\beta\omega a^\dagger a}}{Z} S_z^\dagger D^\dagger(\mu). \tag{2}$$

where $Z = (1 - e^{-\beta\omega})^{-1}$. If you want, try to compute its Husimi function. But in order not to take too much of your time, I'm satisfied with the squeezed vacuum only.

3. **Critical Rabi model.** In this problem we are going to analyze the model proposed in *Phys. Rev. Lett*, **115** 180404 (2015) (arXiv 1503.03090). We start with the Rabi model

$$H = \omega a^\dagger a + \frac{\Omega}{2}\sigma_z - \lambda(a + a^\dagger)\sigma_x. \tag{3}$$

But we will be interested in analyzing this model in the limit

$$\Omega \to \infty, \qquad \lambda \to \infty \qquad \text{but} \quad \frac{\lambda}{\sqrt{\Omega}} = \text{finite}$$

In these cases it is usually convenient to parametrize

$$\Omega = \Omega_0 N^2, \qquad \lambda = \lambda_0 N,$$

where Ω_0 and λ_0 are finite parameters. Then our "thermodynamic limit" corresponds to $N \to \infty$.

We are going to study this problem using a perturbation theory method called the Schrieffer-Wolf transformation (see arXiv 1105.0675 for a thorough discussion). Essentially, we shall consider a transformed Hamiltonian

$$\tilde{H} = U^\dagger H U,$$

where

$$U = \exp\left\{\frac{\lambda}{\Omega}(a + a^\dagger)(\sigma_+ - \sigma_-)\right\}.$$

Note that $\lambda/\Omega = (\lambda_0/\Omega_0)/N$ so this exponential becomes small in the limit of N large.

(a) Show that in the limit of $N \to \infty$ the leading order of the Hamiltonian $\tilde{H}$ is

$$\tilde{H} = \omega a^\dagger a + \frac{\Omega}{2}\sigma_z + \frac{\omega g^2}{4}(a + a^\dagger)^2 \sigma_z$$

where $g = 2\lambda/\sqrt{\omega\Omega}$.

(b) This Hamiltonian is now diagonal in the σ_z basis of the qubit part. Let us then focus on the projection of $\tilde{H}$ on the $|1\rangle$ state, for which it becomes

$$\tilde{H}_- = \omega a^\dagger a - \frac{\Omega}{2} - \frac{\omega g^2}{4}(a + a^\dagger)^2$$

Use the squeezing operator of the previous problem to diagonalize this Hamiltonian. Show that this diagonalization only works provided $g \leq 1$. The fact that something goes wrong at $g = 1$ signals that for $g > 1$ something serious is happening with the ground-state. In case, what happens is that after $g = 1$ the bosonic mode starts to acquire a macroscopic number of photons. It therefore corresponds to a transition to a **superradiant phase**. To treat the problem for $g > 1$ requires some extra work. So I will stop for now in order to not overburden you with work.[3] The interesting result is that for $g > 1$ the ground-state acquires a non-zero expectation value

$$\langle a \rangle = \pm\sqrt{\frac{\Omega}{4g^2\omega_0}(g^4 - 1)} \sim N$$

since $\Omega \sim N^2$. Thus, for $g > 1$ two degenerate ground-states appear with a photon occupation number that scales with $\langle a^\dagger a\rangle \simeq \langle a^\dagger\rangle\langle a\rangle \sim N^2$.

[3] If you are interested in continuing all the way to the end, then what you need is to derive Eq. (4) of *Phys. Rev. Lett*, **115** 180404 (2015) (arXiv 1503.03090). The details on how to do that are given in the supplemental material of the paper.