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Quantum chaos for two interacting particles confined to a circular billiard

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Abstract

We discuss the problem of two quantum particles confined to a circular billiard, interacting through a Yukawa potential and subjected to a weak constant magnetic field. From the statistical analysis of the energy spectrum, we show that the system presents a very interesting oscillation between quantum regular and irregular (chaotic) behavior as a function of the masses ratio of the particles.
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Many-particle problems are of fundamental importance in all branches of Physics, such as celestial mechanics, nuclear and particle physics, etc. For a long time, aspects such as the number of constituents, kind of interactions and type of boundary conditions are well known to strongly determine the different qualitative regimes for the systems dynamics. However, only recently the role played by the ratio between the particles masses has been addressed. In 1D problems, ergodic and non-ergodic motion in classical systems were obtained both by changing the masses ratio of particles with contact interactions [1–4] and by introducing a Yukawa potential between identical particles [5].

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In the quantum case little exists along this line. In 1D, it has been shown that: (i) the level spacing distribution for two interacting particles in a hard-wall billiard is neither purely Poisson or Gaussian orthogonal ensemble (GOE) [6]; and (ii) a transition from regular to irregular spectra takes place for two confined particles interacting by short-range potentials when the masses ratio is changed [7].

For 2D problems, results are even more scarce, with few studies done for square billiards with two interacting particles: (iii) for identical spin $\frac{1}{2}$ particles, both the symmetries of the confining potential and the singlet/triplet crossings lead to a non-GOE spectral distribution [8]; and (iv) if one of the particles is much heavier, acting like a s-wave scatterer, the energy levels of the problem reveal similar features of those seen in Aharonov–Bohm systems [9].

In this contribution, we discuss a new example of the influence of the masses ratio on the statistics of the quantum spectra. We consider two interacting particles under the action of a weak constant magnetic field. However, here we assume as the confinement a *circular* billiard, which as far as we know has not been analyzed in the literature.

In polar coordinates and in arbitrary units the Hamiltonian (with the weak magnetic field perpendicular to the billiard), is given by

$$H = H_0^{(1)} + H_0^{(2)} + V(r_{12}), \quad H_0^{(j)} = \frac{\mathbf{p}_j^2}{2\mu_j} + \frac{BL_z^{(j)}}{2\mu_j}, \quad V(r_{12}) = V_0\,\frac{\mathrm{e}^{-\alpha r_{12}}}{r_{12}}\ . \tag{1}$$

In Eq. (1), $r_{12}=|\vec{r}_1-\vec{r}_2|=\sqrt{r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)}$ is the relative distance, with μ_j and $L_z^{(j)}=\frac{1}{i}\,\partial/\partial\theta_j$ being, respectively, the mass and the angular momentum z-component of particle j. B is the intensity of the external magnetic field. The interaction $V(r_{12})$ (a Yukawa potential) has strength V_0 and length action α. Regarding this last parameter, we can have *short* (e.g., $\alpha=10$) and *long* (e.g., 5×10^{-3}) *range interaction regimes*. While for the first example, the particles feel each other over about $0.25\,r_0$, in the second their interaction extends over the whole billiard. The confinement effect is taken into account by assuming that the wave function must vanishes at the circular billiard boundary $r=r_0$.

The one-particle $H_0^{(j)}$ is a textbook problem, whose eigenstates are given by

$$\psi_{m_jn_j}(r_j,\theta_j) = C_{m_jn_j}J_{m_j}(k_{m_jn_j}r_j)\exp[\pm i m_j\theta_j]\ , \tag{2}$$

where J is the Bessel function and $C_{m_jn_j}=[r_0\sqrt{\pi}J_{m_j+1}(k_{m_jn_j}r_0)]^{-1}$ is the normalization constant. The angular momentum quantum numbers are $m_j=0,1,2,\ldots$. The eigenenergies are given by $E^{(0)}_{m_jn_j}=[k^2_{m_jn_j}\pm Bm_j]/(2\mu_j)$, where $r_0k_{m_jn_j}$ is the n_jth root of the Bessel function of order m_j.

To diagonalize the full two-particles H, we use as a base the complete set $\{\Psi_{\eta_1\eta_2}(r_1,\theta_1;\ r_2,\theta_2)=\psi_{m_1n_1}(r_1,\theta_1)\psi_{m_2n_2}(r_2,\theta_2)\}$, with $\eta_j\equiv(m_jn_j)$. For our purposes in this work, we are going to consider that even for same masses the particles are always distinguishable. The matrix elements are then

$$H_{\eta\chi} = \langle\Psi_{\eta_1\eta_2}|H_0|\Psi_{\chi_1\chi_2}\rangle + \langle\Psi_{\eta_1\eta_2}|V(r_{12})|\Psi_{\chi_1\chi_2}\rangle, \tag{3}$$

where $\eta_j=(m_jn_j)$ and $\chi_j=(q_jl_j)$. The first term in the right-hand side of (3) results trivially in $(E_{m_1n_1}+E_{m_2n_2})\delta_{m_1q_1}\delta_{m_2q_2}\delta_{n_1l_1}\delta_{n_2l_2}$, contributing just to the diagonal elements of H. The second, $I=\langle\Psi_{\eta_1\eta_2}|V(r_{12})|\Psi_{\chi_1\chi_2}\rangle$, has to be calculated. With the help of the transformations $\theta_- = \theta_1 - \theta_2$ and $\theta_+ = \theta_1 + \theta_2$, the integrals in I are simplified and the integration over θ_+ can be performed analytically. After some straightforward analyzes, it is possible to show that I is different from zero only for $L=m_1+m_2=q_1+q_2$, implying that the total angular momentum L is conserved. So, H has a block structure and we can separate the energy levels according to the values of L. For $m_1+m_2=q_1+q_2$, we have (with C a short-hand for the product of the normalization constants)

$$I = C\int_0^{2\pi} \mathrm{d}\theta_- \int_0^{r_0} r_1\,\mathrm{d}r_1 \int_0^{r_0} r_2\,\mathrm{d}r_2(4\pi - 2\theta_-)\cos[(q_1-m_1)\theta_-]$$

$$J_{m_1}(k_{m_1n_1}r_1)J_{m_2}(k_{m_2n_2}r_2)\exp\left[-\frac{\alpha r_{12}}{r_{12}}\right]J_{q_1}(k_{q_1l_1}r_1)J_{q_1}(k_{q_1l_1}r_2). \tag{4}$$

The above integral must be evaluated numerically. Although there are analytical methods [10] to regularize the original Hamiltonian, in order to overcome possible difficulties with the singularity at $\vec{r}_1=\vec{r}_2$, here we use a simpler numerical procedure. The integration over r_1 and r_2 is done in two parts. In the first (second), r_2 (r_1) goes from 0 to $r_1(r_2)-\delta$ and r_1 (r_2) from δ to r_0. So, δ represents how much we are far from the line $\vec{r}_1=\vec{r}_2$, which has to be avoided. The calculations are then performed for decreasing values of δ. For δ between 10^{-4} and 10^{-8} the integral converges, where the very small differences for I within this range is negligible for the seeking precision.

The eigenvalues of the matrix H were determined for a large combination of the parameters V_0, α, $\gamma=\mu_2/\mu_1$ (the masses ratio), and the billiard radius is $r_0=1.0$. Next, we discuss only few of such combinations, but which already help us to figure out what is the main feature responsible for the emergence of quantum chaotic behavior in this two-particles 2D system. We diagonalize 1600×1600 matrices, which for the parameters values considered do assure the convergence of the eigenvalues about four figures for the first 800 levels. We construct two spectrum probabilities: (a) the nearest neighbor spacing distribution $P(s)$, which tells what is the distribution of distances (in terms of the "unfolded" energy s) between two successive energy levels; and (b) the spectral rigidity $\Delta_3(l)$, which basically represents how much (within a normalized energy interval of length l) the cumulative number of states deviates from a straight line. For a detailed explanation of how to calculate these distributions see, for instance, Ref. [11].

Figure 1 shows the results of the level statistics for the case of equal masses ($\gamma=1$), angular momentum $L=0$, potential strength $V_0=10$ and long interaction range $\alpha=5\times10^{-3}$. In the figures, the solid and the dashed lines represent, respectively, the results expecting from a Poisson and GOE level distribution. The Poisson (GOE) distribution is typical for a general quantum system whose corresponding classical system is integrable (chaotic). The histogram (Fig. 1(a)) and dotted line (Fig. 1(b)) represent our numerical calculations, where we have used about 600 levels. We observe from Fig. 1 that the statistical level analysis of equal masses particles confined to a circular billiard agrees very well with the case of integrable systems.

Quantum chaos can be observed for different values for the masses ratio. An example is shown in Fig. 2, where $\gamma=15$ and all the other parameters are the same as those in Fig. 1. The numerical results now agree with the chaotic level statistics.

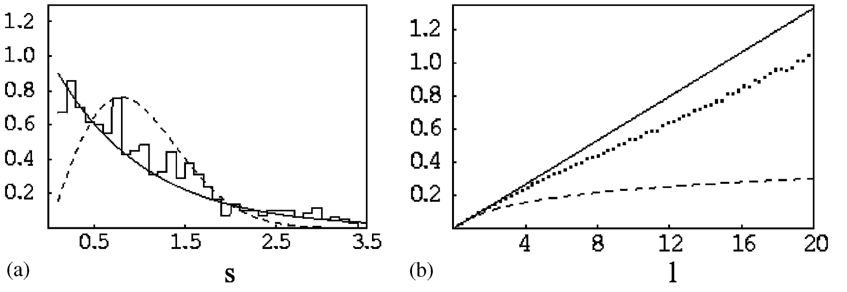

Fig. 1. (a) $P(s)$ and (b) $\Delta_3(l)$ statistics for equal masses ($\gamma = 1$), $V_0 = 10$, $B = 0.1$, and $\alpha = 5 \times 10^{-3}$.

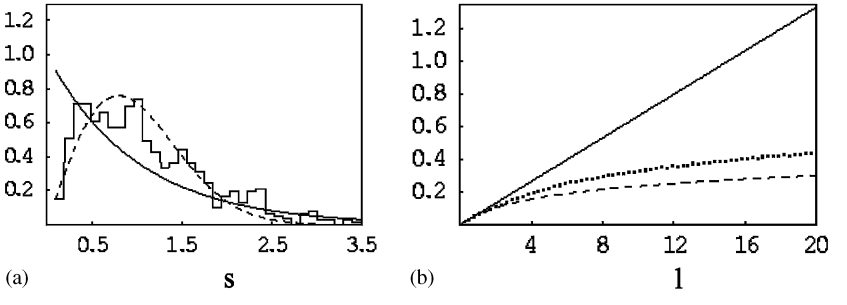

Fig. 2. (a,b) The same as in Fig. 1 but for $\gamma = 15$.

This particular case can be understood from an analogy with the annular billiard [12], known to be chaotic if the inner barrier (a disk) is not placed at the center of the billiard. Since in our example one particle is 15 times heavier, its kinetic energy is much less affected by the mutual interaction. So, it acts just like a potential barrier for the other particle, exactly as in the annular billiard.

From the above we see that, as a function of γ, two limits for the quantum dynamics emerge: the regular for $\gamma = 1$ and the chaotic for $\gamma = 15$. Thus, γ seems to be a control parameter to induce quantum chaos in the system. Similar analysis were performed for many other values of the masses ratio. The results are summarized in Fig. 3, where for the Δ_3 statistics a "bending" coefficient D is plotted for 85 different values of γ. We define $D = [\Delta_3(20) - 0.3]/20$, where $\Delta_3(20)$ is the actual numerically calculated Δ_3 statistics at $l = 20$ and 0.3 is its value in the theoretical chaotic (dashed line) case. If the quantity D approaches zero, the problem can be considered more chaotic. On the other hand, if D increases, it is more regular. Thus, Fig. 3 gives an idea of the "degree of chaoticity" in terms of γ. Although an overall decaying can be observed for γ increasing, so the system tends to be fully chaotic for larger γ's, for some specific values we see peaks, indicating a more regular behavior. These results are quite interesting, *the quantum system oscillates between a more regular and a more irregular dynamics depending on the masses ratio*.

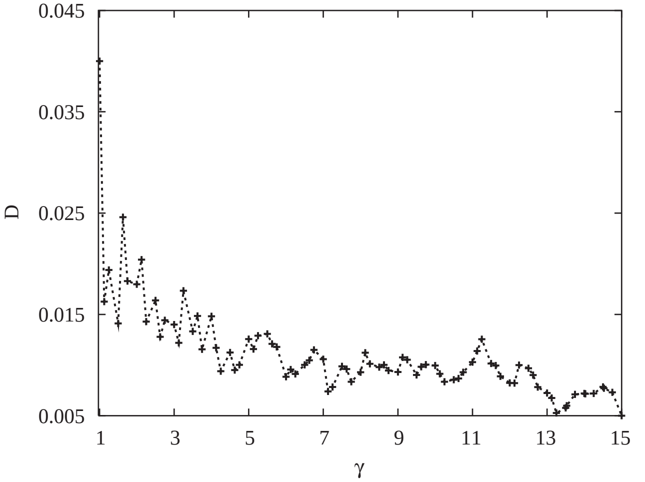

Fig. 3. The "bending" coefficient D as a function of the masses ratio γ for $V_0=10$, $B=0.1$, and $\alpha=5\times10^{-3}$.

Qualitatively, Fig. 3 is in accordance with a recently proposed explanation [7] for the presence (or not) of chaos in few-particles problems. Quantum chaotic behavior may be linked to the ergodicity in the corresponding classical case, which by its turn depends on γ. Indeed, systems like those in Refs. [1–4] are integrable just for a very specific values of the masses ratio. For all other values the systems are ergodic due to the momentum transfer between the particles during collisions.

To conclude, the emergence of quantum chaos is studied for a system of two particles interacting via a Yukawa potential, subjected to a weak constant magnetic field, and confined to a circular billiard. Eigenvalues are determined numerically and the level statistics are studied for different values of the masses ratio $\gamma = \mu_2/\mu_1$. By increasing γ, the system tends to be totally chaotic. However, for some specific values of the masses ratio (within the range $1 \leqslant \gamma < 15$), regular behavior seems to exist. A possible explanation may be related to the break of ergodicity for certain γ's in the underlying classical case, a phenomenon already seen in 1D interacting particle systems. Investigations in this line are under progress and will be reported in the due course.

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