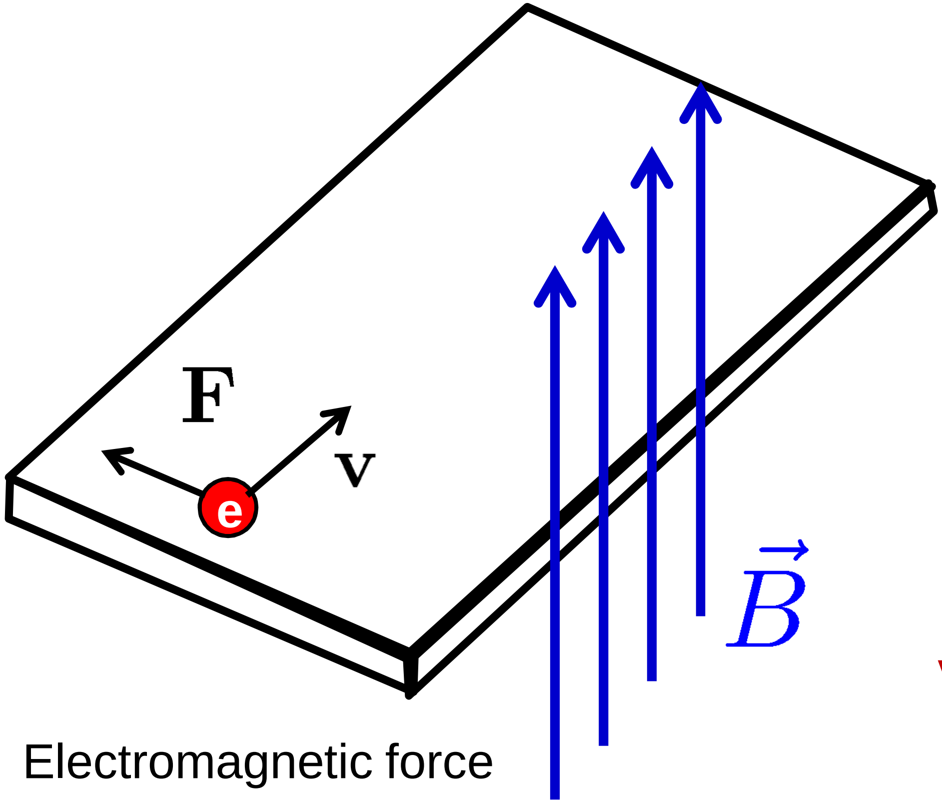
Velocity and Berry curvature in the QHE



Electromagnetic force

$$\mathbf{F} = (-e)\mathbf{E} + (-e)\mathbf{v} \times \mathbf{B}$$

Electric potential and Vector potential.

$$\begin{cases} \mathbf{B} = \nabla_{\mathbf{r}} \times \mathbf{A}(\mathbf{r}) \\ \mathbf{E} = -\nabla_{\mathbf{r}} V(\mathbf{r}) - \dfrac{\partial}{\partial t}\mathbf{A}(\mathbf{r}, t) \end{cases}$$

In the absence of other charges:

$$V(\mathbf{r}) = 0 \Rightarrow \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r}, t)$$

Hamiltonian:

$$H = \frac{(\mathbf{p} + e\mathbf{A}(t))^2}{2m^*} \equiv \frac{\hbar^2 (\mathbf{k}(t))^2}{2m^*}$$

Velocity: $m^*\mathbf{v}(t) = \mathbf{p} + e\mathbf{A}(t) = \hbar\mathbf{k}(t)$

Thus: $\mathbf{v}(\mathbf{k}) = \dfrac{1}{\hbar}\nabla_{\mathbf{k}} H(\mathbf{k})$

Tarefa 20: identity for the velocity

Using:

$$\begin{cases} \mathbf{v}(\mathbf{k}) = \dfrac{1}{\hbar}\nabla_{\mathbf{k}} H(\mathbf{k}) = \dfrac{1}{\hbar}\left(\dfrac{\partial H}{\partial k_x}\mathbf{i} + \dfrac{\partial H}{\partial k_y}\mathbf{j} + \dfrac{\partial H}{\partial k_z}\mathbf{k}\right) \\ H|n, \mathbf{k}(t)\rangle = E_n[\mathbf{k}(t)]|n, \mathbf{k}(t)\rangle \\ i\hbar\dfrac{d}{dt}|n, \mathbf{k}(t)\rangle = i\hbar\dfrac{d\mathbf{k}(t)}{dt} \cdot \nabla_{\mathbf{k}}|n, \mathbf{k}(t)\rangle \\ H|\Psi(t)\rangle = i\hbar\dfrac{d}{dt}|\Psi(t)\rangle \end{cases}$$

Show that:

1) $$\nabla_{\mathbf{k}}(H|n, \mathbf{k}\rangle) = (\nabla_{\mathbf{k}} H)|n\mathbf{k}(t)\rangle + H(\nabla_{\mathbf{k}}|n, \mathbf{k}\rangle)$$

2) $$H(\nabla_{\mathbf{k}}|n, \mathbf{k}\rangle) = i\hbar\nabla_{\mathbf{k}}\left(\frac{d\mathbf{k}(t)}{dt} \cdot \nabla_{\mathbf{k}}|n, \mathbf{k}\rangle\right)$$

3) $$\hbar\mathbf{v}|n\mathbf{k}(t)\rangle = \nabla_{\mathbf{k}}(E_n[\mathbf{k}]|n, \mathbf{k}\rangle) - i\hbar\nabla_{\mathbf{k}}\left(\frac{d\mathbf{k}(t)}{dt} \cdot \nabla_{\mathbf{k}}|n, \mathbf{k}\rangle\right)$$

Tip: Do it by components so you don't get confused!!

Velocity and Berry curvature in the QHE

From the previous result, it follows(*):

$$\mathbf{v}_n(\mathbf{k}) = \langle n, \mathbf{k}(t)|\mathbf{v}|n, \mathbf{k}(t)\rangle = \frac{1}{\hbar}\nabla_{\mathbf{k}} E[\mathbf{k}] + \frac{d\mathbf{k}(t)}{dt} \times \nabla_{\mathbf{k}} \times \langle n, \mathbf{k}|i\nabla_{\mathbf{k}}|n, \mathbf{k}\rangle$$

Remember the definition of the Berry curvature:

$$\mathbf{\Omega}_n(\mathbf{k}) = \nabla_{\mathbf{k}} \times \langle n\mathbf{k}|i\nabla_{\mathbf{k}}|n, \mathbf{k}\rangle$$

and using:

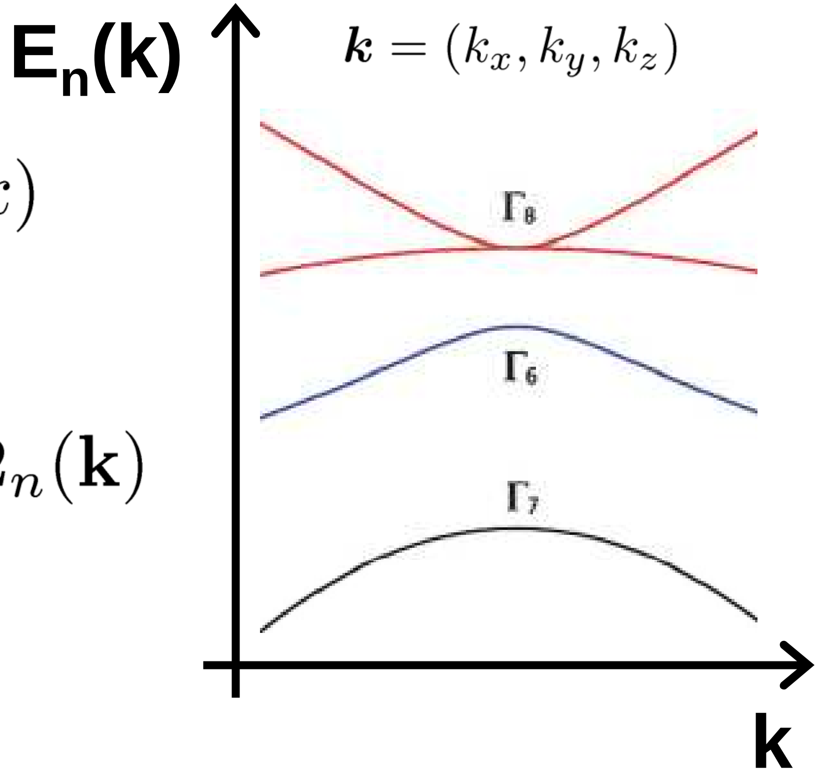
$$\frac{d\mathbf{k}(\mathbf{t})}{dt} = \frac{e}{\hbar}\frac{\partial \mathbf{A}}{\partial t} = -\frac{e}{\hbar}\mathbf{E}$$

we get

$$\boxed{\mathbf{v}_n(\mathbf{k}) = \frac{1}{\hbar}\nabla_{\mathbf{k}} E_n[\mathbf{k}] - \frac{e}{\hbar}\mathbf{E} \times \mathbf{\Omega}_n(\mathbf{k})}$$

(*) Should be on Lista 5!

Hall Conductance and Chern number



Current density(*): $$\mathbf{J} = -e\sum_n \int \frac{d\mathbf{k}}{(2\pi)^2}\mathbf{v}_n(\mathbf{k})f(k)$$

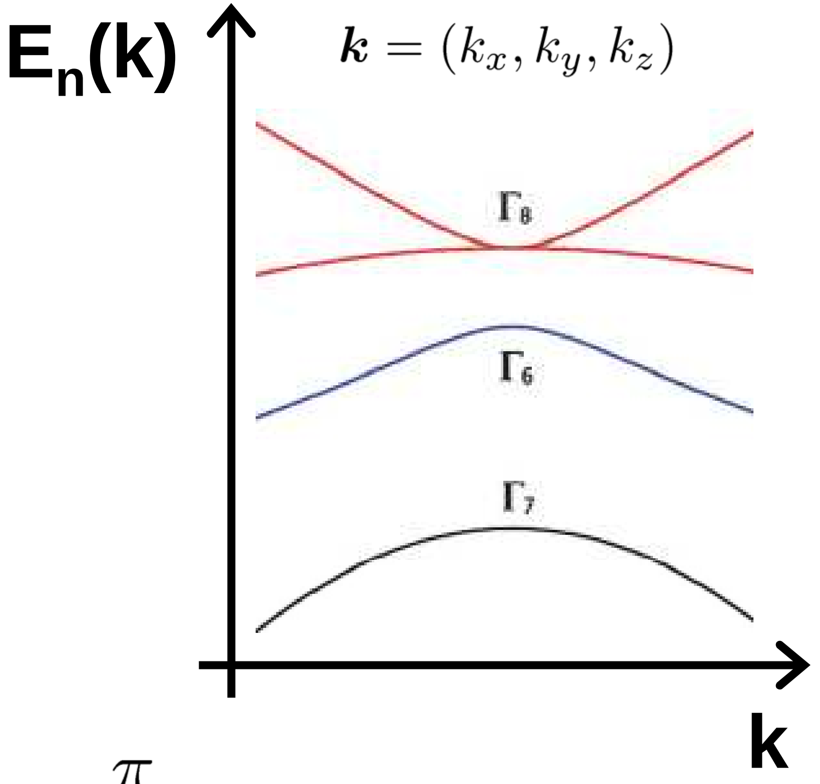
Using: $$\mathbf{v}_n(\mathbf{k}) = \frac{1}{\hbar}\nabla_{\mathbf{k}}E_n[\mathbf{k}] - \frac{e}{\hbar}\mathbf{E}\times\mathbf{\Omega}_n(\mathbf{k})$$

we might calculate the conductance: $$\mathbf{J} = \boldsymbol{\sigma}\cdot\mathbf{E}$$

If we have a gap and N filled levels $$\sum_{n\in\text{filled}}\int \frac{d\mathbf{k}}{(2\pi)^2}\nabla_{\mathbf{k}}E_n[\mathbf{k}]f(k) = 0$$

(*) Quantum version of the usual: $$\mathbf{J} = (-e)n\langle\mathbf{v}\rangle$$

Hall Conductance and Chern number



The conductance can then be calculated: $\mathbf{J} = \boldsymbol{\sigma} \cdot \mathbf{E}$

$$\sigma_{xy} = \frac{e^2}{h}\frac{1}{2\pi}\sum_{n\in\text{filled}}\int_{\text{BZ}} \mathbf{\Omega}_n(\mathbf{k}) \cdot d\mathbf{k}$$

The integral will be carried out in the 1st BZ, which is a torus for the Berry curvature:

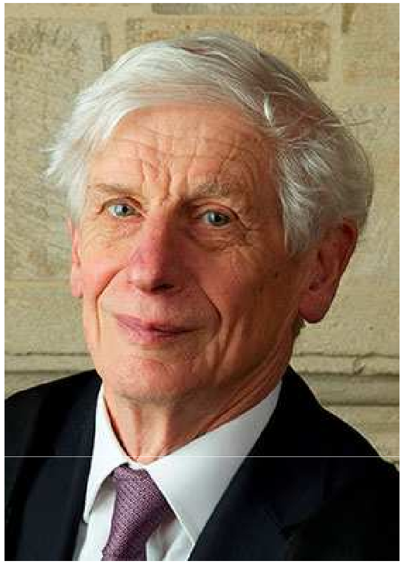
$$\mathbf{\Omega}_n(k_x, k_y) = \mathbf{\Omega}_n(k_x + \frac{\pi}{a}, k_y) = \mathbf{\Omega}_n(k_x, k_y + \frac{\pi}{a})$$

Thus the integral will be 2π (Chern number) and the sum will give the number of filled bands ν :

$$\boxed{\sigma_{xy} = \frac{e^2}{h}\frac{1}{2\pi}2\pi\nu = \frac{e^2}{h}\nu}$$

TKNN invariant: 1982

The Hall conductivity is proportional to a Chern number (Berry-phase-like)



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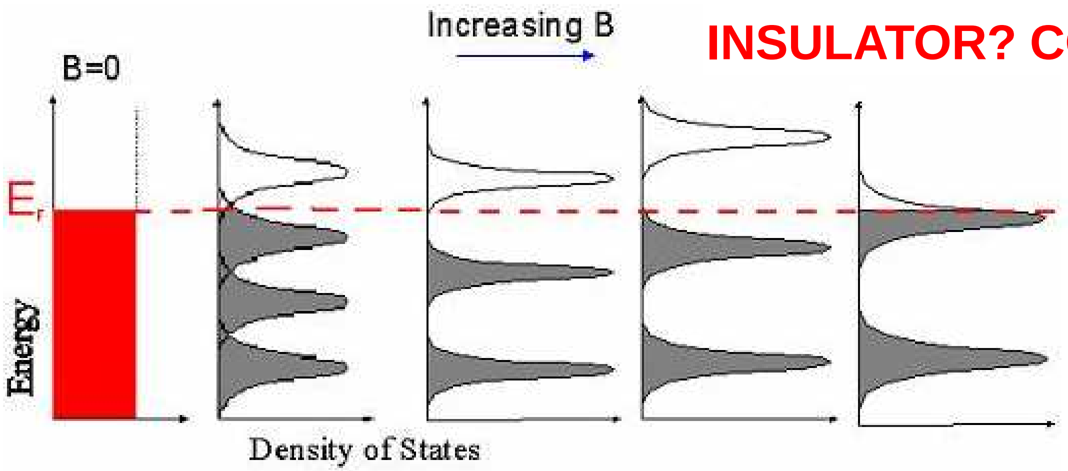
2016

$$\sigma_{xy} = \frac{e^2}{h} \sum_{n<N_F} \frac{1}{2\pi} \iint_{\mathrm{BZ}} \boldsymbol{\Omega}_n(\boldsymbol{k}) \cdot d\boldsymbol{k} \equiv \nu \frac{e^2}{h}$$

Thouless, Kohmoto, Nightingale, den Nijs, *Phys. Rev. Lett*. 49, 405 (1982)

- System is periodic (BZ is a torus in k-space)
- There is an uniform magnetic field in the system.
- Fermi energy lies in a gap with N_F filled bands.

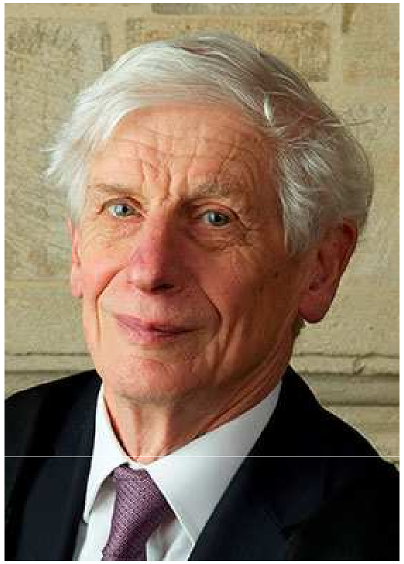


INSULATOR? CONDUCTOR? Neither?

ν = 0,1,2,... : filling factor. Depends only on the topology of the BZ states.

TKNN invariant: 1982

The Hall conductivity is proportional to a Chern number (Berry-phase-like)



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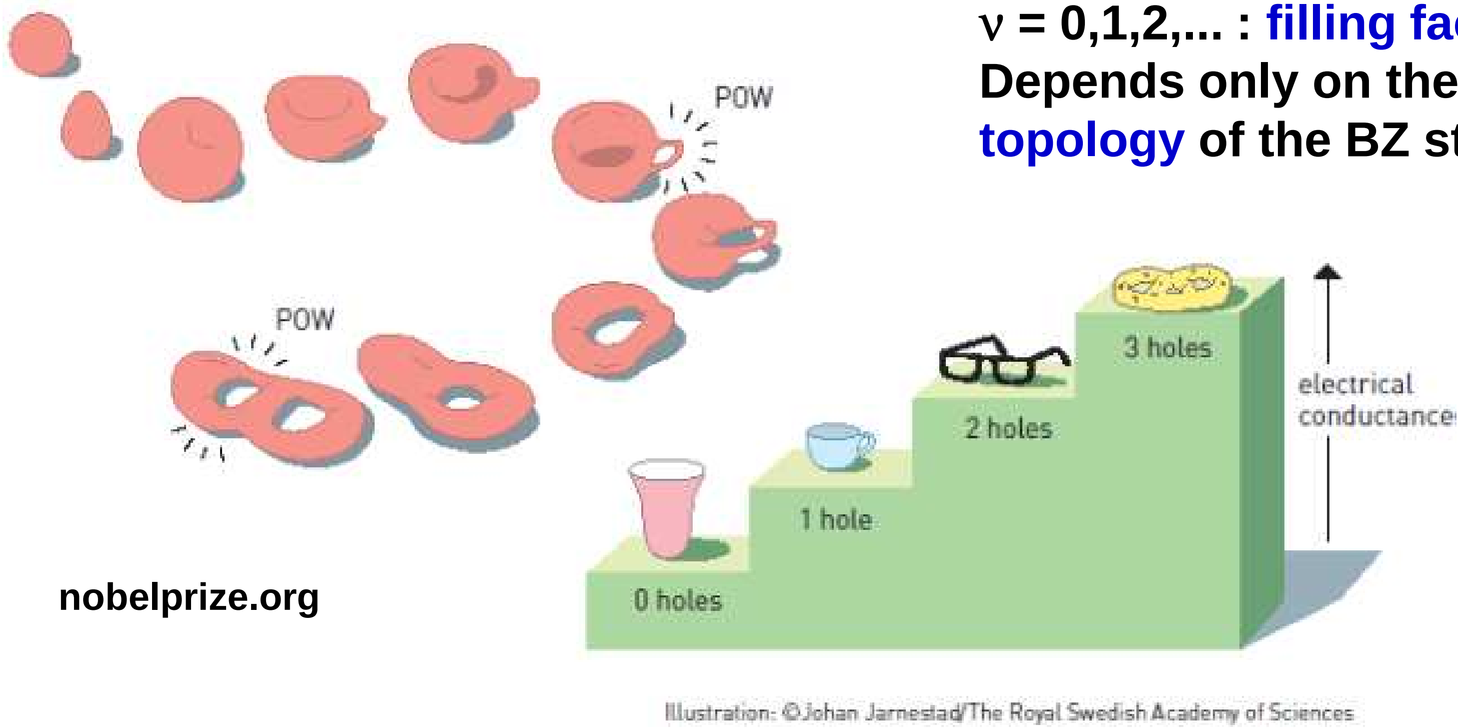

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