

Topological Order and the Kitaev Model

PGF5295-Teoria Quântica de Muitos Corpos em Matéria Condensada

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Abstract

In this notes we intend to discuss the concept of **Topological Order** through the explicit description of an exactly solvable model known as the **Toric Code** which in turn is a particular limit of a larger class of models encompassed by the term **Kitaev Model**.

1 Introduction

Matter can present itself in nature in several different states or phases. The most known of them being gas, liquid and solid. Besides these states of matter we can distinguish other more intricate states that can occur in different situations and scales, for instance: plasmas, Bose-Einstein condensates, quantum spin liquids or superfluids to mention a few.

States of matter are, in general, classified into phases that can be connected to each other via phase transitions. This is, different states of matter can be distinguished by their internal structure or **order**. To visualize this we can think on the example of a solid at some finite temperature where the atoms are arranged in an almost regular pattern whose specific form depends on the solid constituents their interactions and the external conditions such as pressure and temperature. Let us assume that we choose to vary some of this external conditions, say, the temperature, eventually the *crystal order* will undergo a phase transition into a *liquid phase* where the motion of atoms is now less correlated. If we continue raising the temperature the system will again go through a phase transition into a very disordered phase, namely, it will become a gas, where the motion of an atom now hardly depends on the motion of the other constituents.

In this sense, phases of matter could, in principle, be classified by means of phase transitions. Under this scheme different phases of matter have different internal structure or order. One key step in order to develop a general theory that could ultimately classify these phases of matter was the realization that the internal orders of a system are related to the symmetries of its elementary constituents. As a material undergoes a phase transition the internal symmetries of the system change. This is the fundamental idea in what is known as the *Ginzburg-Landau* theory of phase transitions [1, 2, 3] that was originally developed to describe the transition to a superconductor phase of matter by means of a local order parameter and its fluctuations. For a long time it was thought that this theory could describe all phases of matter and their phase transitions.

In the 1980's, a major discovery regarding a system of strongly correlated electrons was made, creating what is known as *2-dimensional electron gas* (2DEG), which when subject to strong magnetic fields at very low temperatures [4] forms a new kind of order. These states are characterized by a quantity coined *filling fraction* that measures the ratio of the electron density and the flux quanta of the external magnetic field that is being applied, this is

$$\nu = \frac{nhc}{eB}, \quad (1)$$

where n is the electron density. Those states for which this quantity is an integer number are called **Integer Quantum Hall** states (IQH) whereas those states for which ν is a fractional number are, correspondingly, called **Fractional Quantum Hall** states (FQH). While the former can be understood from the Landau level structure¹ the latter requires an entirely new theory. The novelty brought by the FQH states is that they show internal orders or “patterns” that do not have any relation with any kind of symmetries (or the breaking of them) and thus cannot be described by the usual Ginzburg-Landau symmetry-breaking scheme. These patterns consist on a highly correlated motion of the electrons around each other such that they do their own cyclotron motion in the first Landau level, an electron always takes integer steps to go around another neighboring electron and they tend to be apart from each other as much as possible (which makes the fluid an incompressible one). It is this global motion pattern that corresponds to the **topological order** in FQH states [5]. Additionally FQH states have a very special feature, as their ground state is degenerate and this degeneracy depends on the topology of the space [6, 7, 8] that can not be modified by perturbations or impurities [5], for instance the Laughlin state with $\nu = 1/q$ has a degeneracy of q^g where g is the genus of the manifold the system is embedded into; thus, only a change in the internal pattern or topological order (implying a change on ν for the cited state) can induce a change in the ground state degeneracy, since topological order is a property of the ground state of the system.

The features of topologically ordered systems are not restricted to the topology dependent ground state degeneracy. The low energy excitations of such systems exhibit characteristic properties such as the fractionalization of the charge and anyonic statistics. In particular, for the FQH states, which arise in systems whose constituents are electrons each one with charge e , the excitations carry a charge that is a fraction of e . Although this might seem to be disconnected with the topological order at first glance, it is closely related to the degeneracy as it can be shown that fractionalization implies the degeneracy of the ground state [9, 10]. Another feature involving the quasi particle excitations of such systems is that they obey exotic statistics. In 3 spatial dimensions it is known that the quantum states of identical particles behave either as bosons or fermions under an exchange of a pair. Nevertheless, in two dimensional systems, such as the FQH states, there are new possibilities for quantum statistics that interpolate between those of bosons and fermions. Under an exchange of two quasi-particles the quantum state can acquire an overall phase $e^{i\theta}$, where the special cases $\theta = 0, \pi$ correspond to the bosonic and fermionic statistic respectively. The **statistical angle** θ can take different values, and the particles obeying these generalized statistics are called **anyons** [11, 12, 13, 14, 15, 16].

Another phenomenon related to topological order can be exemplified through the excitations arising at the edges of the FQH liquids. The excitations on the bulk of the system are gapped, meaning there is a finite energy difference between the ground state and the elementary excited states. Nevertheless, in the boundary of the system (the FQH liquid) gapless excitations arise². The structure of the edge excitations depend on the bulk topological order, as for different types of order in the bulk there are different edge excitations structures, that can be understood as surface (edge) waves propagating along the border of the FQH liquid [17, 18].

Thus, FQH liquids are very different from any other state of matter because they ordered in a different way, they exhibit topological order. Hence there is a need for a general theory of topological phases and consequently the need for a mathematical framework that could ultimately characterize and classify these topological phases of matter. In the past few years there has been a major interest on the study of these phases of matter via a detailed analysis of exactly solvable

¹The motion of a single electron subject to a magnetic field consists in a circular orbit which is quantized (due to the wave-particle duality) in terms of the wavelength of the electron, so if an electron takes n steps to go around a circular orbit we say it is in its n^{th} Landau level.

²A *gapless* excitation is such that its energy goes to zero as its momentum goes to zero, or equivalently, only an infinitesimal amount of energy is required to go from the ground state to an excited state.

lattice models which exhibit the features of having **topological order**. The simplest example is the so called **Toric Code** model introduced by A. Kitaev in [19] which is constructed as a many body interacting system defined over a 2-dimensional lattice. It exhibits the features of a topologically order system as its ground state is 4-fold degenerate when the lattice is embedded on the surface of a Torus, hence part of its name. The degeneracy is protected from local perturbations that come as the elementary excitations of the model. These elementary excited states can be interpreted as quasi-particle anyonic excitations located at the vertices and faces of the lattice, they display both bosonic and fermionic statistics when braided among themselves. This model can be interpreted as a particular lattice gauge theory [20] where the gauge group is the abelian \mathbb{Z}_2 group [21]. Furthermore, for any finite group G , in [19] Kitaev introduces a more general class of models called **Quantum Double Model** (QDM) defined through a Hamiltonian that is written as a sum of mutually commuting projectors [22, 23, 24, 25]. The elementary excitations of this models are anyons whose fusion and braiding properties depend on the specific choice of the group G giving rise to the possibility of having non-abelian anyons that can be used to implement a fault-tolerant quantum computation process [26, 27, 28, 29], where unitary transformations are obtained by the braiding of anyons and the final measurement is performed by the joining of pairs of excitations. Moreover, a large class of topological orders were identified by the systematic construction of the so called **string-net** models [18, 30] where it is shown that each topological phase is associated to a *fusion category*. The QDM being a subclass of these models as shown in [31].

2 Kitaev Model

Also known as the **Honeycomb model** the model is defined on an hexagonal two dimensional lattice, $1/2$ spin degrees of freedom live on the vertices of the lattice. The dynamics is governed by the following Hamiltonian

$$H = -J_x \sum_{\text{x-links}} \sigma_j^x \sigma_k^x - J_y \sum_{\text{y-links}} \sigma_j^y \sigma_k^y - J_z \sum_{\text{z-links}} \sigma_j^z \sigma_k^z, \quad (2)$$

where J_x , J_y and J_z are the model's free parameters. The interactions are split into three types depending on the kind of link as shown in Fig. 1. It is through the variation of these free parameters that the parameter space is explored and we will focus on a particular limit coined **topological limit**.

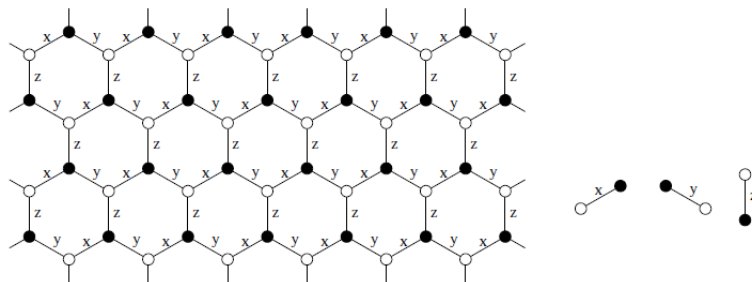


Figure 1: Honeycomb lattice, the x, y and z bonds are explicitly shown

As an aside we will like to say that the model can be exactly solved by mapping the spins into four majorana fermions. This transformation makes the Hamiltonian a quadratic one which makes an exact solution possible. We refer the reader to [32] for the details of such solution. The phase diagram of the model exhibits both gapped and gapless phases. In particular, the case when $|J_z| \gg |J_x|, |J_y|$ the model lies in a gapped phase. In such a phase the correlations

decay exponentially with distance, which means that the quasi-particles cannot interact directly. Nevertheless, the excitons interact topologically via a braiding procedure. To analyze the topological properties of this gapped phase we will first map the model into a discrete gauge theory Hamiltonian via perturbation theory. This is done since the ground state and the low energy excitations are easily studied using this new Hamiltonian.

2.1 Perturbation Theory

The Hamiltonian is now written as:

$$H = H_0 + V = -J_z \sum_{\text{z-links}} \sigma_j^z \sigma_k^z - J_x \sum_{\text{x-links}} \sigma_j^x \sigma_k^x - J_y \sum_{\text{y-links}} \sigma_j^y \sigma_k^y. \quad (3)$$

Let us start considering the case when $|J_x| = |J_y| = 0$, notice that the ground state of this sector consists on a highly degenerate dimer gas where both spins on a z-bond are aligned. Therefore, each pair can be regarded as a single spin. The equivalence is shown on the first two figures in Fig.2.

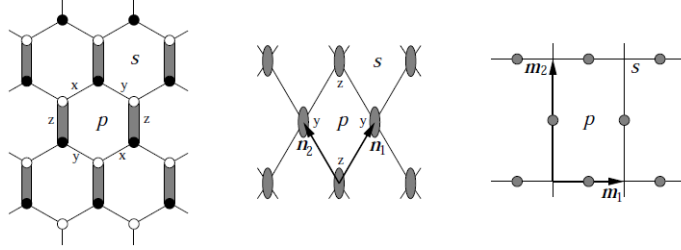


Figure 2: On the left, the z-bonds are mapped into a single spin degree of freedom as shown in the middle figure. On the right each effective spin is mapped to the edges of a square lattice.

The idea is to find an effective Hamiltonian that acts on the Hilbert space of effective spins \mathcal{H}_{eff} living on the vertices of a square lattice. This can be done using the Green function formalism as follows. Consider the map $\Gamma : \mathcal{H}_{eff} \rightarrow \mathcal{H}$ that sends the effective Hilbert space into the ground state subspace of H_0 . The map Γ simply doubles each spin, namely $\Gamma |m\rangle = |mm\rangle$. The eigenvalues of H_{eff} are supposed to be the energy levels of H that originate from H_0 . These energy levels can be defined to be the poles of the Green function $G(E) = \Gamma^\dagger (E - H)^{-1} \Gamma$ which is an operator acting on the effective Hilbert space. The effective Hamiltonian can be written as $H_{eff} = E_0 + \Sigma(E_0)$, where $\Sigma(E_0)$ is called the self-energy. $\Sigma(E_0)$ is computed via a Dyson series. Let $G_0(E) = (E - H_0)^{-1}$ be the unperturbed Green function for the excited states of H_0 . Then the self energy is given by:

$$\Sigma(E) = \Gamma^\dagger (V + VG_0(E)V + VG_0(E)VG_0(E)V + \dots) \Gamma \quad (4)$$

Then, set $E = E_0$ and compute $H_{eff} = E_0 + \Sigma$ in the zeroth order: $H_{eff}^{(0)} = E_0$. The following orders are calculated until a non constant term is found, namely:

$$H_{eff}^{(1)} = \Gamma^\dagger V \Gamma = 0. \quad (5)$$

$$H_{eff}^{(2)} = \Gamma^\dagger V G_0 V \Gamma = - \sum_{\text{x-links}} \frac{J_x^2}{4J_z} - \sum_{\text{y-links}} \frac{J_y^2}{4J_z} = -N \frac{J_x^2 + J_y^2}{4J_z}. \quad (6)$$

$$H_{eff}^{(3)} = \Gamma^\dagger V G_0 V G_0 V \Gamma = 0. \quad (7)$$

$$H_{eff}^{(4)} = \Gamma^\dagger V G_0 V G_0 V G_0 V \Gamma = - \frac{J_x^2 J_y^2}{16J_z^3} \sum_p Q_p, \quad \text{where: } Q_p = \sigma_{i_1}^y \sigma_{i_2}^y \sigma_{i_3}^z \sigma_{i_4}^z \quad (8)$$

Thus the effective Hamiltonian up to an additive constant reads

$$H_{eff} = -\frac{J_x^2 J_y^2}{16|J_z^3|} \sum_p Q_p, \quad (9)$$

where the spin configuration corresponds to that of the middle Fig.2.

2.2 The Toric Code

The Hamiltonian of eq.(9) can be further transformed into a more familiar form [19, 33, 34]. This procedure corresponds to first mapping the square lattice with spins living at its vertices to a new square lattice where the spin degrees of freedom now live at the edges (c.f. Fig.2). Notice that the plaquettes of the first square lattice become plaquettes and vertices of the new square lattice. As a consequence the effective Hamiltonian now is given by:

$$H_{eff} = -J_{eff} \left(\sum_{\text{vertices}} Q_s + \sum_{\text{plaquettes}} Q_p \right), \quad (10)$$

where the effective coupling parameter is that of eq.(9). Secondly, let us apply the following unitary transformation on the effective Hamiltonian (10)

$$U := \prod_{\text{horizontal links}} X_j \prod_{\text{vertical links}} Y_k, \quad (11)$$

such that the Hamiltonian 10 becomes:

$$H'_{eff} = U H_{eff} U^\dagger = -J_{eff} \left(\sum_{\text{vertices}} A_v + \sum_{\text{plaquettes}} B_p \right), \quad (12)$$

where the vertex and plaquette operators are respectively defined by:

$$A_v = \bigotimes_{i \in \text{star}(v)} \sigma_i^x = \sigma_{i_1}^x \otimes \sigma_{i_2}^x \otimes \sigma_{i_3}^x \otimes \sigma_{i_4}^x, \quad B_p = \bigotimes_{j \in \partial p} \sigma_j^z = \sigma_{j_1}^z \otimes \sigma_{j_2}^z \otimes \sigma_{j_3}^z \otimes \sigma_{j_4}^z, \quad (13)$$

where the index $i \in \text{star}(v)$ runs over all four edges around an specific vertex v and $j \in \partial p$ stands for all four edges in the boundary of a plaquette p of the lattice, as shown in Fig. 3 and the Pauli matrices are given by:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (14)$$

The algebra of the vertex and plaquette operators is induced by the one of Pauli operators, recalling the commutation (and anticommutation) relations

$$[\sigma^x, \sigma^x] = 0 \quad (15)$$

$$[\sigma^z, \sigma^z] = 0 \quad (16)$$

$$\sigma^x \sigma^z = -\sigma^z \sigma^x. \quad (17)$$

These expressions fix the commutation relations of the operators B_p and A_v . It is clear that all B_p commute with each other because of eq.(16), similarly all A_v commute with each other since

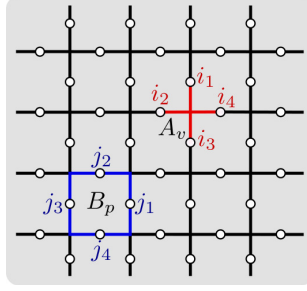


Figure 3: A piece of the square lattice; The spin degrees of freedom lie on the edges of the lattice and are represented by hollow circles. The sites being acted by the *Plaquette Operator* B_p and the *Vertex Operator* A_v are explicitly shown.

eq.(15) holds. The commutation relation between A_v and B_p is slightly less trivial. However, it is straightforward to show that they commute, i.e.

$$[A_v, B_p] = 0, \quad (18)$$

for any vertex v and plaquette p in \mathcal{L} .

Since the Pauli matrices are involutory, i.e., $(\sigma^x)^2 = (\sigma^y)^2 = (\sigma^z)^2 = \mathbb{1}$, where $\mathbb{1}$ is the 2×2 identity matrix; it is easy to see that the operators A_v and B_p also square to identity, namely:

$$A_v^2 = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} = B_p^2. \quad (19)$$

Now, let us consider an eigenstate $|\psi_v\rangle$ for the Vertex Operator A_v and $|\psi_p\rangle$ an eigenstate of the Plaquette Operator B_p , such that:

$$A_v |\psi_v\rangle = a |\psi_v\rangle \quad B_p |\psi_p\rangle = b |\psi_p\rangle,$$

because of the involutory nature of both operators we notice that

$$(A_v)^2 |\psi_v\rangle = \mathbb{1} |\psi_v\rangle = a^2 |\psi_v\rangle \quad (20)$$

$$(B_p)^2 |\psi_p\rangle = \mathbb{1} |\psi_p\rangle = b^2 |\psi_p\rangle, \quad (21)$$

therefore, the only values a and b can take are ± 1 . Given the Hilbert space on the edges is spanned by the states $|\phi_1\rangle$ and $|\phi_{-1}\rangle$ that can be chosen to be the eigenstates of the σ^z Pauli operator, this is, we choose a representation for the basis states where

$$|\phi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |\phi_{-1}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (22)$$

such that the action of the Pauli matrices on the states is the following:

$$\begin{aligned} \sigma^x |\phi_1\rangle &= |\phi_{-1}\rangle & \sigma^x |\phi_{-1}\rangle &= |\phi_1\rangle \\ \sigma^z |\phi_1\rangle &= |\phi_1\rangle & \sigma^z |\phi_{-1}\rangle &= -|\phi_{-1}\rangle. \end{aligned}$$

2.3 Ground States

To find the ground state $|\Psi_0\rangle$ of H_{eff} , we need to find the condition for the energy to be minimum in Eq. (12), or equivalently (because of the negative sign in front of each term in Eq. (12)) we need to maximize the eigenvalues of each A_v and B_p operators. Given that their eigenvalues are ± 1 , it is clear that a lower bound on the energy is obtained when the ground state $|\Psi_0\rangle$ satisfies

$$B_p |\Psi_0\rangle = |\Psi_0\rangle \quad \forall p, \quad \text{and} \quad A_v |\Psi_0\rangle = |\Psi_0\rangle \quad \forall v, \quad (23)$$

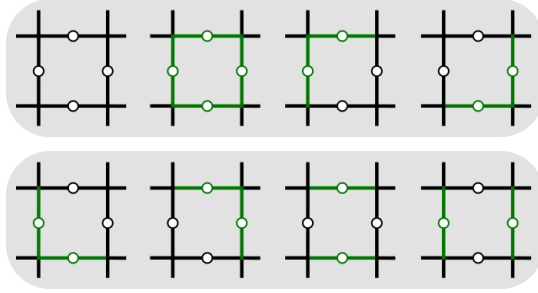


Figure 4: All possible vortex-free spin configurations for a single plaquette are shown, where we consider an arbitrary plaquette p and only down spin $|\phi_{-1}\rangle$ states are green colored.

therefore any state that fulfills these two conditions will be a ground state of the system. Leaving for the ground state energy $E_0 = -(N_p + N_v)$, where N_p and N_v stand for the number of plaquettes and vertices present on the lattice.

Consider a plaquette operator acting on the states living on the boundary of an arbitrary plaquette p , let us call this state $|\phi_p\rangle$, such that

$$B_p |\phi_p\rangle = \pm |\phi_p\rangle, \quad (24)$$

notice that the eigenvalue of B_p will depend on the configuration of the four spin degrees of freedom living on the edges along the plaquette p . In particular, if the plaquette operator has eigenvalue 1 when applied to a particular state we say the state is *vortex-free*. Recalling that the plaquette operator is a tensor product of Pauli matrices σ^z whose action on the basis states $\{|\phi_1\rangle, |\phi_{-1}\rangle\}$ is given by:

$$\sigma^z |\phi_1\rangle = |\phi_1\rangle, \quad \sigma^z |\phi_{-1}\rangle = -|\phi_{-1}\rangle, \quad (25)$$

it is clear that the state will be a *vortex-free* one for configurations with an even number of $\{|\phi_1\rangle, |\phi_{-1}\rangle\}$ states. We show all possible *vortex-free* states for a single plaquette in Fig.(4) where we adopt a graphical representation in which the edges containing a $|\phi_{-1}\rangle$ state are green colored.

Let us now look at the action of the vertex operator on a given state, we know the vertex operator acts with σ^x on the 4 edges adjacent to the vertex in question, the action of this Pauli operator on the basis states $\{|\phi_1\rangle, |\phi_{-1}\rangle\}$ is given by

$$\sigma^x |\phi_1\rangle = |\phi_{-1}\rangle, \quad \sigma^x |\phi_{-1}\rangle = |\phi_1\rangle, \quad (26)$$

this is, the vertex operator action consists on flipping the states on each edge from $|\phi_1\rangle$ to $|\phi_{-1}\rangle$ and viceversa, for instance let $|v\rangle = |\phi_1\rangle_{i_1} \otimes |\phi_1\rangle_{i_2} \otimes |\phi_1\rangle_{i_3} \otimes |\phi_1\rangle_{i_4}$, where i_1, i_2, i_3 and i_4 are the four edges adjacent to an arbitrary vertex, the action of the vertex operator on this state is given by:

$$A_v \left| \begin{array}{c} \bullet \\ | \text{---} \text{---} \text{---} \text{---} \\ \bullet \end{array} \right\rangle = \left| \begin{array}{c} \bullet \\ | \text{---} \text{---} \text{---} \text{---} \\ \bullet \end{array} \right\rangle, \quad (27)$$

where the edges holding $|\phi_{-1}\rangle$ states are green colored. In this sense, the vertex operator can be understood as a local gauge transformation, equivalently we say that two configurations that are related to each other by the action of a vertex operator are *gauge equivalent*. In Fig.(5) we show several actions of the vertex operator on arbitrary states of a given vertex.

Let us go back to the analysis of the ground state $|\Psi_0\rangle$ for the Hamiltonian defined in Eq.(12), the first condition on Eq.(23) involving the plaquette operator restricts the configurations of the

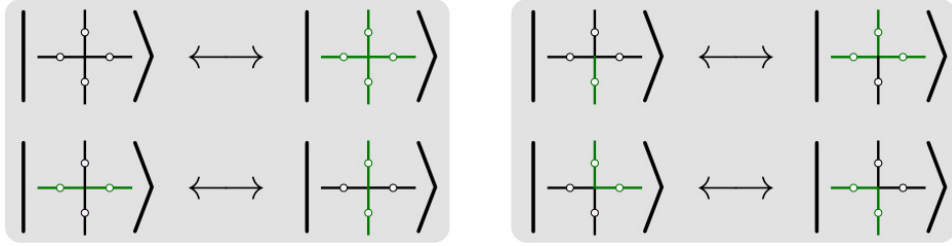


Figure 5: Some illustrative actions of the Vertex Operator A_v are shown using the graphical representation for the states, where the colored edges hold down spin $|\phi_{-1}\rangle$ states. We say that two states that are related by the action of the vertex operator have *gauge equivalent* configurations.

ground states as being *vortex-free*, such as the ones shown in Fig.(4), therefore we can write the most general ground state as [35, 21]

$$|\Psi_0\rangle = \sum_{s: B_p|s|=|s| \forall p} c_s |s\rangle, \quad (28)$$

where $|s\rangle$ is a *vortex-free* state and the ground state $|\Psi_0\rangle$ is a linear combination of all *vortex-free* configurations. The second condition in Eq.(23) involving the vertex operator tells us that any two configurations on a given vertex that are related to each other by the action of a vertex operator will appear in the ground state with the same weight, this is, all coefficients c_s are required to be equal. Hence, a ground state of the toric code is an equal-weight superposition of vortex free configurations.

The explicit form of such state is given by:

$$|\Psi_0^v\rangle = \prod_{v \in \mathcal{L}} \frac{1}{\sqrt{2}} (\mathbb{1} + A_v) \bigotimes_{l \in \mathcal{L}} |\phi_1\rangle_l, \quad (29)$$

where the product runs over all vertices in the lattice and by the tensor product we mean that each edge l of the lattice \mathcal{L} carries a $|\phi_1\rangle$ state. The state $|\Psi_0\rangle$ is indeed a ground state of the Hamiltonian (12), i.e.

$$A_v |\Psi_0\rangle = B_p |\Psi_0\rangle = |\Psi_0\rangle, \quad \forall v, p \in \mathcal{L}, \quad (30)$$

as it can be shown by using the commutation relations between vertex and plaquette operators $[A_v, A_{v'}] = 0 = [A_v, B_p]$.

Consequently, any product of vertex (plaquette) operators will act trivially on $|\Psi_0^v\rangle$, equivalently, any operator $Z(\gamma)$ or $X(\gamma^*)$ where γ and γ^* define contractible loops on the direct and dual lattice, respectively, will act on $|\Psi_0^v\rangle$ in a trivial way. This ground state is interpreted as being a **Loop Gas**, where the loops are the result of the action of vertex operators on the initial configuration $\bigotimes_{l \in \partial p} |\phi_1\rangle_l$. Each term on the product of eq.(29) will produce a combination of loops defined on the dual lattice and the ground state is a linear combination of all resulting states, some of them are shown in Fig.(6).

The red closed loops on the graphical representation stand for the action of products of A_v operators on the state with all edges holding $|\phi_1\rangle$, it is in this sense that the ground state of the Toric Code can be interpreted as a **Loop Gas**, containing all possible contractible loops that could be defined on the lattice. From now on we consider the lattice to be embedded on the surface of a Torus. We know the torus has two non-contractible loops. Thus, the operators $X(C'_1)$, $X(C'_2)$, $Z(C_1)$ and $Z(C_2)$ can be defined and will not be made of products of vertex or plaquette operators, correspondingly. The action of the $X(C'_1)$ and $X(C'_2)$ operators is to create new ground states since they commute both with the vertex and plaquette operators [19, 36, 37]. Consider the operator $X(C'_1)$ where the path C'_1 winds the Torus along a horizontal

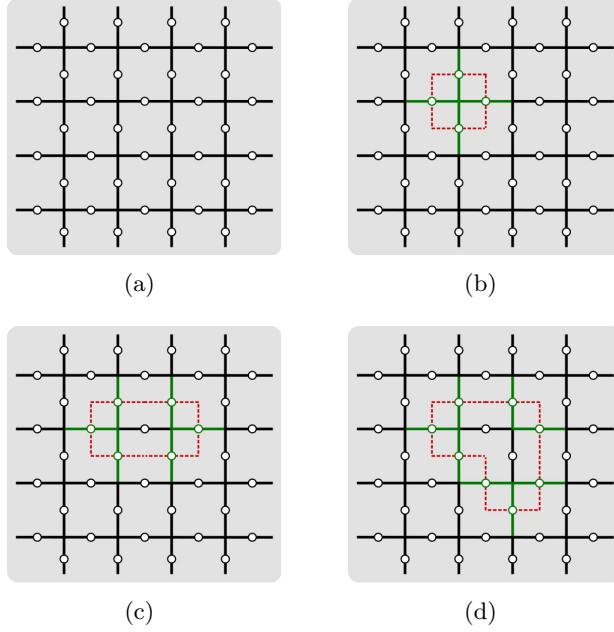


Figure 6: Some illustrative constituents of the ground state $|\Psi_0^v\rangle$ are shown, where (a) corresponds to the first term in the expansion of Eq.(29), (b) corresponds to a term in this expansion for which there is a single A_v acting on the $|\phi_1\rangle$ states around v , in (c) and (d) we show terms that include the action of several vertex operators.

non-contractible loop on the dual lattice, the action of this operator on the ground state $|\Psi_0^v\rangle$ is given by:

$$\begin{aligned}
X(C'_1)|\Psi_0^v\rangle &= \prod_{v \in \mathcal{L}} \frac{1}{\sqrt{2}} (\mathbb{1} + A_v) X(C'_1) \bigotimes_{l \in \mathcal{L}} |\phi_1\rangle_l \\
&= \prod_{v \in \mathcal{L}} \frac{1}{\sqrt{2}} (\mathbb{1} + A_v) \bigotimes_{l \notin C'_1} |\phi_1\rangle_l \bigotimes_{l' \in C'_1} |\phi_{-1}\rangle_{l'} = |\Psi_1^v\rangle,
\end{aligned} \tag{31}$$

where in the first line we used the commutation relation $[X(C'_1), A_v] = 0$, and the action of the winding operator $X(C'_1)$ on the basis states $|\phi_1\rangle$ along the non-contractible loop C'_1 consists on changing them into $|\phi_{-1}\rangle$. Clearly this is still a ground state under the action of any vertex operator A_v , and it is straightforward to show that

$$B_p |\Psi_1^v\rangle = |\Psi_1^v\rangle. \tag{32}$$

by noticing that the plaquette operators acting on edges being crossed by C'_1 will act trivially, since loop C'_1 will necessarily cross any plaquette through two of its edges, therefore the configuration of the degrees of freedom on the plaquettes will be those shown in Fig.(4), this is, vortex free ones.

Thus, we have shown that this new state is a ground state of the Toric Code Hamiltonian. Again it can be interpreted as a **Loop Gas**, some of its constituents are depicted in Fig.(7). Notice that the particular path defined by C'_1 along the dual lattice is not relevant as long as it winds the Torus through a non-contractible one, since all possible deformations of the path are already contained in the Loop Gas. Since this dual path cannot be written as a product of vertex operators, the ground state $|\Psi_1^v\rangle$ cannot be written in terms of $|\Psi_0^v\rangle$ Likewise, acting with $X(C'_2)$ on $|\Psi_0^v\rangle$ we can create another ground state, namely:

$$X(C'_2)|\Psi_0^v\rangle = |\Psi_2^v\rangle. \tag{33}$$

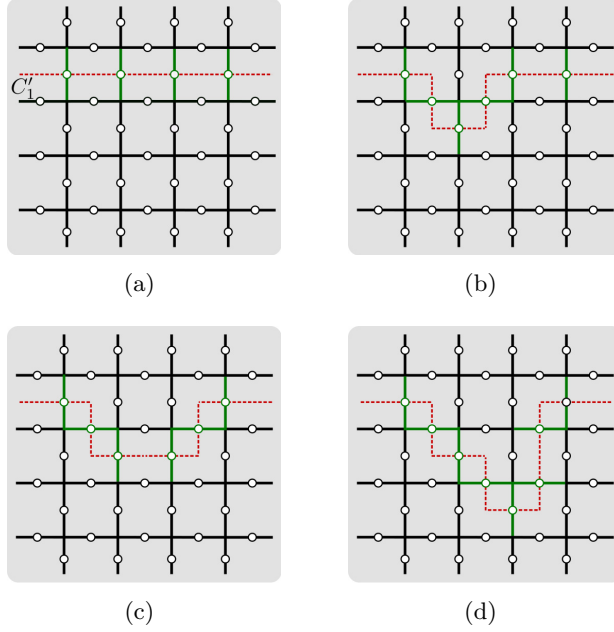


Figure 7: The path C'_1 winds the torus along a non-contractible loop, the operator $X(C'_1)$ defined on this path transforms the degrees of freedom that lie on C'_1 , the state $|\Psi_1^v\rangle$ is composed of all such transformations that can be gotten by the action of vertex operators in \mathcal{L} .

Also by acting with $X(C'_1)$ and $X(C'_2)$ simultaneously on $|\Psi_0^v\rangle$ we create another ground state, i.e.:im

$$X(C'_1)X(C'_2)|\Psi_0^v\rangle = |\Psi_{1,2}^v\rangle. \quad (34)$$

Thus, $|\Psi_0^v\rangle$, $|\Psi_1^v\rangle$, $|\Psi_2^v\rangle$ and $|\Psi_{1,2}^v\rangle$ are the four ground states of the Toric Code, and their existence is guaranteed as long as the lattice is embedded on a Torus. Note that if we allow the lattice to be embedded on a more general surface with genus g more ground states can be constructed, depending on the number of homotopically inequivalent non-contractible loops that can be defined on such surface. Hence the dependence of the ground state on the topological properties of the surface the model is defined in.

The toric code Hamiltonian can be seen as a lattice gauge theory [20], in this framework the degrees of freedom living on the edges of the lattice correspond to \mathbb{Z}_2 valued gauge degrees of freedom, the vertex operator corresponds to a gauge transformation, and since it commutes with the plaquette operator implies an overall gauge invariance. It is from this equivalence to a gauge theory that the excitations of the toric code that come from violating one of the conditions in Eq.(23) are commonly known as *charges* and *fluxes* as we will see in the next section.

3 Elementary Excitations

The elementary excitations of the Toric Code are the result of violating the constraints on Eq.(23). From Eqs.(20) and (21) we know the eigenvalues of the vertex and plaquette operators when acting on an eigenstate can only be ± 1 . Thus, an elementary excitation corresponds to an eigenstate $|\Psi\rangle \in \mathcal{H}$ such that for some vertex and/or plaquette operator it has eigenvalue -1 . This can be seen as follows, an operator that *creates* an excitation at some vertex and/or plaquette of the lattice, is such that it anticommutes with the vertex (plaquette) operator in question.

3.1 Vertex Excitations: Charges

Consider the following state

$$|\Psi_i^z\rangle = \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \sigma_i^z \otimes \cdots \otimes \mathbb{1} |\Psi_0\rangle, \quad (35)$$

where $|\Psi_0\rangle$ is some of the four ground states of the Toric Code Hamiltonian. This new state is no longer a ground state of the model for the condition in Eq.(23) involving the vertex operator A_v does not hold anymore since the two vertex operators A_{v_1} and A_{v_2} sharing the i -th (see Fig.(8)) edge do not commute with the σ^z that is acting on it, this is:

$$\begin{aligned} A_{v_1, v_2} |\Psi_i^z\rangle &= A_{v_1, v_2} \sigma_i^z |\Psi_0\rangle \\ &= -\sigma_i^z A_{v_1, v_2} |\Psi_0\rangle \\ &= -\sigma_i^z |\Psi_0\rangle \\ &= -|\Psi_i^z\rangle \end{aligned}$$

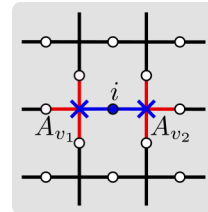


Figure 8: We denote the action of the σ^z operator on i by coloring it blue and the presence of an excitation of this type is represented by a blue “x”.

In this sense we say that the Pauli operator σ_i^z **creates** two excitations located at vertices v_1 and v_2 . The energy of such excited state is then

$$E_v = -(N_p + N_v - 2). \quad (36)$$

Let us now consider an operator of the form:

$$Z(\gamma) = \bigotimes_{j \in \gamma} \sigma_j^z, \quad (37)$$

where γ stands for an open path on the direct lattice, although we do not write it explicitly it is understood that this operator is acting as identity on the lattice sites that are not part of the path γ . This **string operator** as defined in Eq.(37) will anticommute with the vertex operators acting on the endpoints of γ thus creating a pair of localized excitations which henceforth will be called **charges**, as a consequence the excitations can be “moved” by simply extending the path γ on the lattice, the energy of the excited state does not change since the number of vertex operators being affected by the path operator is the same, eventually we can think on closing the path thus creating a loop, in this case no pair of excitations is created since such operator can be written as the product of B_p ’s where the plaquettes are those enclosed by the path γ , in §2.3 we have already shown that this kind of operators act trivially on a ground state.

3.2 Plaquette Excitations: Fluxes

Likewise, the excitations coming from the violation of the first constraint in Eq.(23) involving the plaquette operator B_p are created by a **string operator** given by

$$X(\gamma^*) = \bigotimes_{i \in \gamma^*} \sigma_i^x, \quad (38)$$

where γ^* defines a path in the dual lattice such as the one shown in Fig.(9). The state given by the action of this operator on a ground state $|\Psi_0\rangle$ defined as:

$$|\Psi^x(\gamma^*)\rangle = X(\gamma^*) |\Psi_0\rangle, \quad (39)$$

is an excited state of the Hamiltonian carrying two localized excitations on the plaquettes where the string operator $X(\gamma^*)$ has its endpoints. As in the case of charges the energy of this excited

state depends only on the number of plaquettes that are being affected by the path operator $X(\gamma^*)$ and since they always come in pairs the energy of the state is given by:

$$E_p = -(N_p - 2 + N_v). \quad (40)$$

Note that the states $|\Psi^z(\gamma)\rangle$ and $|\Psi^x(\gamma^*)\rangle$ and their corresponding energy do not depend on the path itself, to illustrate this consider the state $|\Psi^x(\gamma^*)\rangle$ that has a pair of fluxes living on two plaquettes p_1 and p_2 , there are many paths on the dual lattice that join the aforesaid plaquettes, analogously there is an equal number of path operators $X(\gamma^*)$ that create this pair of fluxes. We say that the state $|\Psi^x(\gamma^*)\rangle$ (and its energy) rather depend on the homotopy class³ of the path γ^* whereas the operators that map a given ground state into the excited states depend on the path itself.

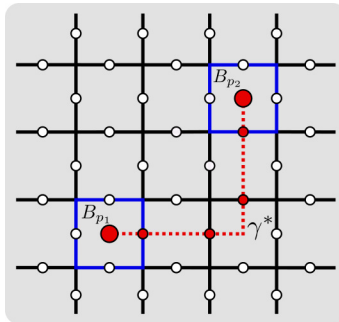


Figure 9: The operator $X(\gamma^*)$ creates two localized excitations (red circles) at the plaquettes p_1 and p_2 , we denote the action of the σ^x operators on the edges of the lattice by coloring them red.

3.3 Fusion and the Dyonic Excitation

Let us now consider the case when we have pair of disconnected charges, by this we mean they were created by means of two path operators $Z(\gamma_1)$ and $Z(\gamma_2)$ as depicted in Fig.(10), this state is the result of the action of two string operators $Z(\gamma_1)$ and $Z(\gamma_2)$ on a ground state $|\Phi_0\rangle$, i.e.

$$|\Psi^z(\gamma_{1,2})\rangle = Z(\gamma_1)Z(\gamma_2)|\Phi_0\rangle, \quad (41)$$

consider this as being our initial state, now we decide to connect these two paths by extending either γ_2 or γ_1 through an additional path γ_3 , on doing so note that the excitations that in principle were sitting at the endpoints of both γ_1 and γ_2 annihilate with themselves as the resulting operator $Z(\gamma_1 + \gamma_2 + \gamma_3) = Z(\gamma_1)Z(\gamma_2)Z(\gamma_3)$ now commutes with the vertex operators acting at v_1 and v_2 . Thus, we say that the charge excitations are their own antiparticles, as when we **fuse** them together they cancel each other:

$$e \times e = 1, \quad (42)$$

here e stands for the presence of a charge excitation at some vertex and 1 for the absence of excitations at the given vertex.

In a similar manner, we can imagine the process of creating two disconnected flux excitations each one sitting at arbitrary plaquettes p_1 and p_2 and then fusing the fluxes by extending one of the dual paths. The result is an state where the two of the original fluxes were annihilated. This induces the following **fusion rule**:

$$m \times m = 1, \quad (43)$$

³Two maps are said to be homotopically equivalent or to belong to the same homotopy class if one can be continuously deformed into the another.

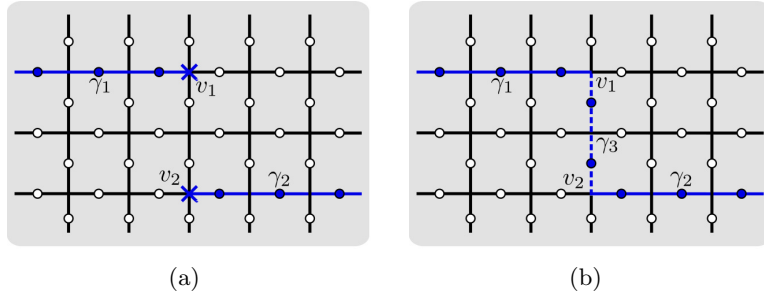


Figure 10: In (a) we show the initial configuration of the excited state $|\Psi^z(\gamma_{1,2})\rangle$. Then in (b) the path γ_3 is an extension of either γ_1 or γ_2 .

where m stands for the presence of a flux on a particular plaquette. Moreover, consider a state that is the outcome of a simultaneous application of string operators $Z(\gamma)$ and $X(\gamma^*)$ such that they share one (or both) of its endpoints (see Fig.(11)), this is, there are operators σ^x and σ^z being applied at the edge where the paths end. As discussed in §3.1 and §3.2 this process creates a charge and a flux which in this case are regarded as a single excitation called **Dyon**.

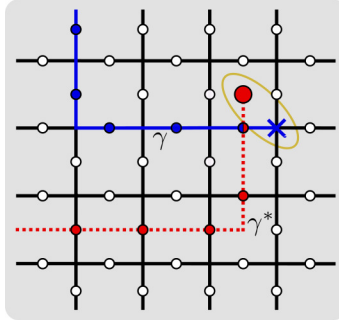


Figure 11: The dyonic excitation is the result of fusing a charge together with a flux.

This can be interpreted as the **fusion rule**:

$$e \times m = m \times e = \epsilon. \quad (44)$$

From these three fusion rules we can deduce the following ones:

$$\epsilon \times e = m, \quad \epsilon \times m = e, \quad (45)$$

and for a matter of completeness we add the trivial fusion rules

$$1 \times 1 = 1, \quad 1 \times e = e \times 1 = e, \quad 1 \times m = m \times 1 = m, \quad 1 \times \epsilon = \epsilon \times 1 = \epsilon. \quad (46)$$

3.4 Braiding Statistics

We now study the statistical behavior of the elementary excitations of the model, namely the charge e , the flux m and the dyon ϵ . This is, what happens to the overall state when we move one of the excitations around another. To begin with, consider an state that has a pair of localized charge excitations such as the initial configuration in Fig.(10); instead of fusing these two charges together consider a process by which we **exchange** their position by applying two path extensions $Z(\gamma'_1)$ and $Z(\gamma'_2)$ (see Fig.(12)).

Thus, the initial state $|\Psi^z(\gamma_{1,2})\rangle$ is the result of applying string operators $Z(\gamma_1)$ and $Z(\gamma_2)$ on a ground state, as in Eq.(41), now, the exchange procedure is given by the action of operators

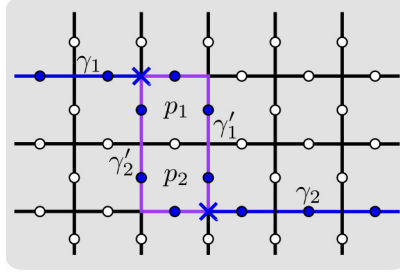


Figure 12: The braiding of two charges by transporting them along the paths γ'_1 and γ'_2 (in purple). Note that both these extension paths can be written as products of plaquette operators acting at p_1 and p_2 .

$Z(\gamma'_1)$ and $Z(\gamma'_2)$, such that the final state would be:

$$|\Psi^z(\gamma_{1,2,1',2'})\rangle = Z(\gamma'_1)Z(\gamma'_2)|\Psi^z(\gamma_{1,2})\rangle, \quad (47)$$

the two additional paths form a loop enclosing two plaquettes, namely p_1 and p_2 , thus we can write them as the product of plaquette operators, i.e., $Z(\gamma'_1)Z(\gamma'_2) = B_{p_1}B_{p_2}$, therefore the state defined in Eq.(47) can be written as:

$$|\Psi^z(\gamma_{1,2,1',2'})\rangle = B_{p_1}B_{p_2}|\Psi^z(\gamma_{1,2})\rangle = |\Psi^z(\gamma_{1,2})\rangle, \quad (48)$$

in the last equality we note that the action of B_{p_1} and B_{p_2} on the initial state is trivial since there are no flux excitations sitting at the mentioned plaquettes. Hence, the final state of the system is the same as the initial one. The exchange of two charges leaves the state invariant, thus signaling the bosonic nature of charges relative to themselves. A similar argument holds for the exchange of two plaquette excitations or fluxes.

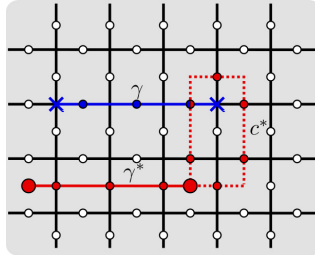


Figure 13: The braiding of a flux around a charge is described, the initial state contains a pair of charges at the endpoints of the path γ (in blue) and a pair of fluxes at the endpoints of the dual path γ^* (in red). Then one of the fluxes is taken around one charge by means of the operator $X(c^*)$.

On the other hand, consider the exchange of a charge and a flux. The initial state, containing a pair of charges and a pair of fluxes, is given by

$$|\Psi_i\rangle = Z(\gamma)|\Psi^x(\gamma^*)\rangle = Z(\gamma)X(\gamma^*)|\Psi_0\rangle, \quad (49)$$

where the paths γ and γ^* are those depicted in Fig.(Fig.(13)), to move one of the fluxes around one of the charges we extend the dual path γ^* by means of another path on the dual lattice which we call c^* . Thereby the final state will be given by

$$\begin{aligned} |\Psi_f\rangle &= X(c^*)|\Psi_i\rangle = X(c^*)Z(\gamma)|\Psi^x(\gamma^*)\rangle \\ &= -Z(\gamma)X(c^*)|\Psi^x(\gamma^*)\rangle \\ &= -|\Psi_i\rangle, \end{aligned} \quad (50)$$

where the minus sign in the third line comes from the fact that the operators $Z(\gamma)$ and $X(c^*)$ anti commute at the edge where they cross each other and, in the last equality we used $X(c^*)|\Psi^x(\gamma^*)\rangle = |\Psi^x(\gamma^*)\rangle$ since the paths γ^* and c^* belong to the same homotopy class. Comparing Eq.(49) with Eq.(50) we notice that the process of moving one of the fluxes around one charge induces an overall -1 phase to the state. The overall phase gained can be interpreted as a Aharanov-Bohm effect. The existence of anyons is at the root of the topologically dependent ground state degeneracy as argued in [14] and both are manifestations of the underlying **Topological Order** of the model.

4 Final Remarks

In these notes we intended to discuss the so called topological phases of matter by a detailed description of the simplest exactly solvable model that has **Topological Order** at zero temperature, namely, the **Toric Code** that was introduced by A. Kitaev in [19], which is a many body interaction model defined on a bi-dimensional lattice and described by a Hamiltonian that consists on a sum of two kinds of mutually commuting projectors and are written in terms of the well known **Pauli Matrices**. The model is exactly solvable and we explicitly construct the ground state of the model which is a long range entangled gas of loops. Moreover, we show the topological origin of the ground state degeneracy by means of operators whose domains are determined by paths on the (dual) lattice, such that, when the lattice is embedded on the 2-torus, these operators can wind the torus along two homotopically inequivalent *non-contractible* loops. Such operators are responsible of the creation of more linearly independent ground states. The elementary excitations of the model arise as violations of the ground state conditions of Eq. (23) for the vertex and plaquette operators, thus they come in two species, labeled **charges** and **fluxes**. Such excitations are localized quasi-particles as they are created at specific vertices and/or plaquettes, they can be moved through the lattice sites at will without an energy cost. More importantly, they can be braided around each other exhibiting their statistical features. We show that the charges and fluxes are bosons when exchanged between themselves, while the braiding of a charge around a flux (or a flux around a charge) is a fermionic process as the state gains an overall -1 sign after an exchange, the same holds for the braiding of two dyons. In [19] a more general class of models is also introduced, coined **Quantum Double Models**. These are interpreted as the Hamiltonian formulation of a lattice gauge theory for a finite and discrete symmetry group G . The model is defined through a Hamiltonian that is a sum of quasi-local commuting projectors, thus making the model a exactly solvable one. The elementary excitations of the QDM are obtained through the action of **Ribbon Operators** that are a generalization of the **String Operators** that create the excitations of the Toric Code. We do not include them in this manuscript as they are not within the scope of the present work and we refer the reader to Kitaev's original work [19], and [33, 38]. The QDM's had been widely studied during the recent years from various points of view. They exhibit topological order. Their ground state is protected from local perturbations which makes them suitable for implementing quantum error correcting codes; furthermore, the quasi particle excitations have anyonic statistics whose braiding is used to implement the quantum gates that could ultimately lead to a quantum computation process [19, 16, 29, 28, 26, 39].

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