Little Introduction to Statistics of Lattice Particles and String-Net Condensed

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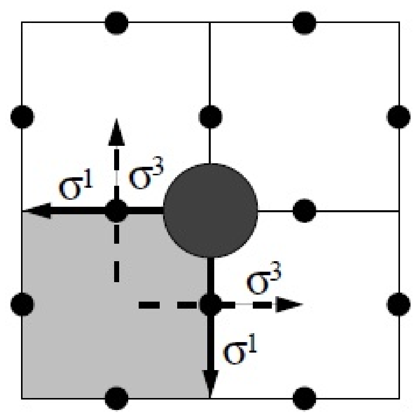
1 Introduction

The purpose is to introduce statistics for lattice particles. We propose fermion particles as emergent in an exactly soluble model for bosons; excitations for the electrical and magnetic operator are bosons and they always come in pairs. Having in mind the toy model Toric code to introduce the structure, as the creation operator has a string like appearance with the particles at the ends always created in pairs. Fermionic excitations come with a non trivial gauge arising from a 'deconfined' phase from a new property, topological order. The main characteristics are the stable ground state degeneracy and non trivial particles statistics. We scale up to more general models which can show fractional statistics as anyons. We present the hopping operator and we will briefly introduce to same further concept of the string net condensed arising from developing the mathematic of tensor category for the statistics with more degrees of freedom.[1]

2 Statistics

The Toric code was first proposed by Kitaev in the '90 as a simple lattice model with topological aspects. The ground state presents degeneracy depending on the genus of the variety analyzed. We have a topological qft, so we can analyze the partition function as only dependent on the topology, for $T = 0$.
It is important to underline that in these models, we don't have a real time evolution, we proceed as exploring different possible choices in the configuration space with the same energy, an adiabatic process. This means we can move the excitations on the lattice without any energy cost.

Building quantum field theory with spins living on the links of a lattice we configure the degrees of freedom for the gauge group choosing the group algebra, for the cyclic group $\mathbb{Z}_2$ we recover the Toric code (on a 2d lattice model). Elementary excitations are bosons (both for the electrical and magnetic operators) while compositions of the two are fermions and anyons (depending on the algebraic group). We can view the excitations as strings, gauge fluctuations on the lattice, with fermions carrying the corresponding gauge charges at the ends.



Here in the figure, we show an excitation composed by a bound state of the electric excitation, a charge (dark circle) and a flux one (the shaded square). The continuous line shows the corresponding electric arrow with hopping operator σ^3, while the dotted one for the flux has hopping operator σ^1; both the operators commute, while between them they anti-commute.

We can differentiate the action with a contribution just local and a topological (global, as the homotopy class) .

$$S = S_{loc} + S_{top} \tag{1}$$

Statistics on the lattice is dealing with the exchange of different particles, for indistinguishable particles it means a unitary transformation in the many bodies wave function or in the path integral formulation the multiplication for a phase. This phase can be well measured in experiments for the *Aharonov-Bohm effect*, linked to the winding number, the possibility for a charged particle q to circle around the magnetic flux Φ (for spinors

$U = (-1)^F$).

$$U(2\pi) = exp(-i2\pi J) = exp(iq\Phi) \tag{2}$$

We have a link for the angular momentum J to the Berry phase θ.
$J = m - \frac{q\theta}{2\pi}$
The amplitude of a closed path must be of the form $e^{iS_{top}} = (\pm 1)^n$ where n is the number of particle exchanges that occurs along the path. The exchange of particles is called braiding.
We can think of the statistics as comparing two paths which only differ for the exchange of two particle, their difference must be then $e^{iS_{top}} = (\pm 1)$, the statistical phase, distinguishing between fermions and bosons. We take a two particle state $|r_1, s_1>$, and consider a product of hopping amplitudes along a lattice path which exchanges the particles. (Each amplitude moves just a particle to the neighboring site.)[1]

$$\begin{aligned} &< r_n, s_n|H|r_{n-1}, s_{n-1} >< r_{n-1}, s_{n-1}|H|r_{n-2}, s_{n-2} > \\ &\quad ... < r_3, s_3|H|r_2, s_2 >< r_2, s_2|H|r_1, s_1 > \end{aligned} \tag{3}$$

The Hilbert space do not depend on the order (hard-core identical particles).

$$|i_1, i_2, ...i_n >= |i_2, i_1, ...i_n > \tag{4}$$

The general Hamiltonian for this system is

$$H = \sum_{<ij>} (t_{ij} + t_{ji}) \tag{5}$$

where t_{ij} are "hopping operators" with the property that

$$t_{ij}|j, i_1, ..., i_{n-1} >\propto |i, i_1, ..., i_{n-1} > \tag{6}$$

For bosons or fermions we can consider the $H = -t\sum_{<ij>}(c_i^\dagger c_j + c_j^\dagger c_i)$ with operators c_i obeying the commutator (anti-) rule from the statistics. The hopping operators do obey the phase rule

$$t_{il}t_{ki}t_{ij} = e^{i\theta}t_{ij}t_{ki}t_{il} \tag{7}$$

for any three hopping operators, where j, k, l are (distinct) neighbors of i (ordered in the clockwise direction in the case of 2 dimensions). The orientation convention in 2 dimensions is necessary for the anyonic case.
So the two paths are

$$|i, j, \text{swapped} >= (t_{jj'})(t_{j'p}...t_{qr})(t_{il}t_{lm}...t_{nj'}) \tag{8}$$

$$(t_{j'j})(t_{rs}...t_{ti})|i, j, ... > \tag{9}$$

Remembering the locality condition

$$[t_{ij}, t_{kl}] = 0 \tag{10}$$

if i, j, k, l are all different, we recover the phase difference after same reordering operation

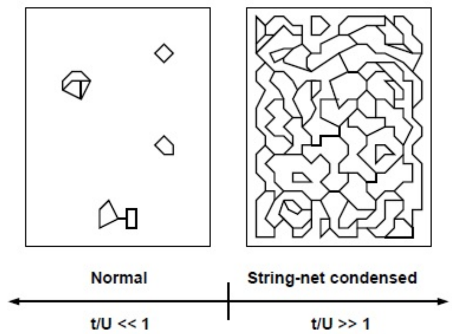
$$|i, j, \text{swapped} >= e^{i\theta}|i, j, \text{unswapped} >$$

To soothing things out we remember the *Spin Statistic theorem*: the rotation of one particle of 2π is the same as the double exchange of two particles, we check what this means in respect of the dimensionality. For the 1D case it is ambiguous, we can't discern between the exchange and the interaction between particles for the additional phase.
In the 2D scenario the matrix $U \in \mathbb{SO}(2)$, the angular moment J is TP invariant just for $\frac{\theta}{2\pi} = J = 0, \pi$; in the contrary we deal with fractional spin: anyons.
While in 3D every loop can be contracted to a point, so the matrix $U \in \mathbb{SO}(3)$ is connected to the identity possessing eigenvalues ± 1 for rotations of 2π.
It is worth noting by that the theorem implies the existence of anti-particles.

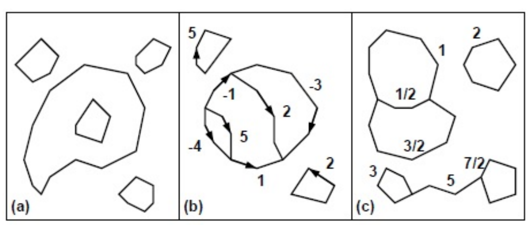
In the non Abelian case we can split and fuse different kind of particles described by $a \times b = \sum_c N^c_{ab} c$, where $N \in \mathbb{N}$, creating a Hilbert topological space of finite dimension. Although it looks like representation theory, we are not considering the direct sum between vector spaces.

3 String-Net Condensed

The laws of Nature seem to be composed of identical particles, gauge interactions, Fermi statistic, chiral fermions and gravity; in the quest of a fundamental structure giving rise to these phenomena, we can search for an emergent behavior of a theory without putting by hands the ingredients. As a partial solution we discuss the string-net condensed where the gauge bosons and fermions can be described as excitations of boson model, in a context of new collective modes emerging from a new phase of matter.[3]
We have discussed in the previous chapter the existence of pair excitations which can be pictorially described by strings, we can think them as a network with oriented, labeled edges. With rules for the kind of strings that meet at a point as the branching rules, we have a well defined model and we can describe its dynamic.



A typical Hamiltonian is divided in a potential, or tension of the string, and a kinetic part $H = UH_u + tH_t$.[2]
When the potential dominates for $U \gg t$ we have few strings in the ground state, and in our previous model it corresponds to $H_U = \sum_i \sigma_i^x$ the electrical energy.
While in the opposite limit the kinetic energy fills the ground state with fluctuating strings corresponding to a model where the magnetic energy dominates. After the transition the strings can be described by a six indexed object F^{ijk}_{lmn} (ruled by tensor category), the conditions of branching, fusion, braiding and statistics put constraints on the products.



In the figure above we present some examples for different gauge group and the rule s to satisfy for a well defined model:
(a) $\mathbb{Z}_2$ it is called the loop gas, we don't need to specify the orientation of the edge; (b) $\mathbb{U}(1)$ we need an orientation for the paths, the label is linked to the flux value and it can show non closed loops, while in the vertices we have the Gauss rule as condition; (c) $\mathbb{SU}(2)$ no orientation is needed, but we have an additional condition, the triangle rule, for the vertices.

In this models we can build bosons as fluctuations of the strings, while fermions arise at their end points, emerging naturally.
Although at the moment there isn't a way to describe chiral fermions, Weak interaction and gravity, as a promising model these fundamental ingredients are missing, but it is still interesting as it looks at gauge theories not as something geometrically fundamental but a by product of coupling constants choices.[3]

References

[1] Michael Levin and Xiao-Gang Wen. Fermions, strings, and gauge fields in lattice spin models. 02139, 2003.

[2] Michael a. Levin and Xiao Gang Wen. String-net condensation: A physical mechanism for topological phases. *Physical Review B - Condensed Matter and Materials Physics*, 71(4):1–21, 2005.

[3] Michael Aaron Levin and Xiao-Gang Wen. A Unification of light and electrons through string-net condensation in spin models. 02139, 2004.